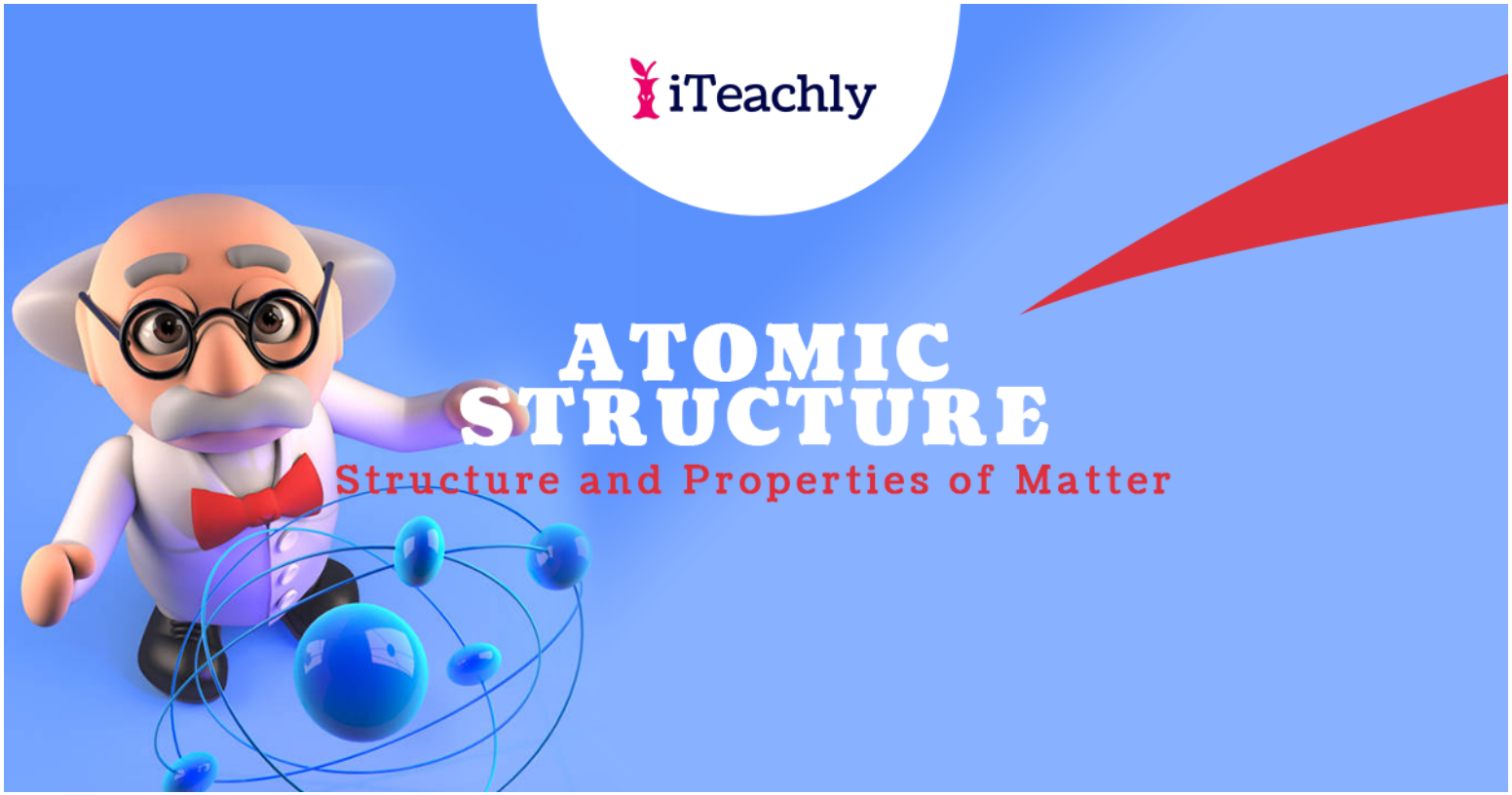
# Atomic Physics

## Chapter 1

## Basic Properties of Atom

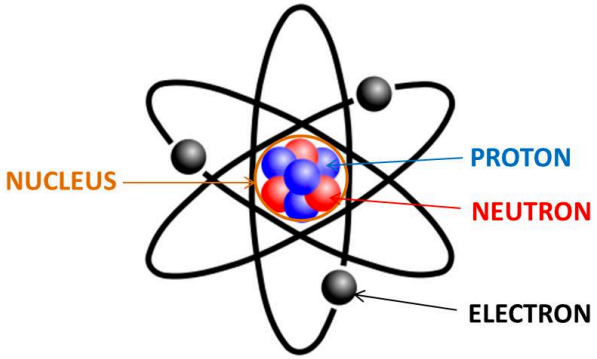


# What is an atom

An atom is the smallest unchangeable component of a chemical element.

1. Unchangeable means in this case by chemical means

2. Moderate temperatures: $kT$ < eV




Mass range: $1.67\times10^{-27}$ to $4.52\times10^{-25}$ kg

Electric charge: zero (neutral), or ion charge

Diameter range: 62 pm (He) to 520 pm (Cs)

Components: Electrons and compact nucleus of protons and neutrons

# The mass of an atom

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**Atomic mass unit (AMU):**

1u: 1/12 of the mass of a neutral carbon atom with nuclear charge 6 and mass number 12

**Mass number (A):**

The total number of protons and neutrons in nucleus

**Mole (mol):**

1 mol is the quantity of a substance that contains the same number of particles (atoms or molecules) as 0.012 kg of carbon $^{12}C$.

1 mol of atoms or molecules with atomic mass number A AMU has a mass of A grams.

# The mass of an atom

## The relation between 1u and $N_A$

$$1u = \frac{1}{N_A} = 1.660539040(20) \times 10^{-27}\ \mathrm{kg}$$

## Electronvolt

$$1\ \mathrm{eV} = 1.602176565(35) \times 10^{-19}\ \mathrm{C} \times 1\ \mathrm{V}$$
$$= 1.602176565(35) \times 10^{-19}\ \mathrm{J}$$

## Mass-energy equivalence

$$E = mc^2$$

## 1u transfer to eV

$$1\ \mathrm{u} = 931.478 \times 10^6\ \mathrm{eV}/c^2$$
$$= 931.478\ \mathrm{MeV}/c^2$$

# The mass of an atom

## The mass of electron:

$$\begin{aligned} m_e &= 9.10938356(11) \times 10^{-31}\ \text{kg} \\ &= 5.48579909070(16) \times 10^{-4}\ \text{u} \\ &= 0.5109989461(31)\ \text{MeV} \end{aligned}$$

## The mass of proton:

$$\begin{aligned} m_p &= 1.672621898(21) \times 10^{-27}\ \text{kg} \\ &= 1.007276466879(91)\ \text{u} \\ &= 938.2720813(58)\ \text{MeV} \end{aligned}$$

## The mass of neutron:

$$\begin{aligned} m_n &= 1.674927471(21) \times 10^{-27}\ \text{kg} \\ &= 1.00866491588(49)\ \text{u} \\ &= 939.5654133(58)\ \text{MeV} \end{aligned}$$

# Avogadro's Number



**Avogadro's number is a Bridge from macroscopic to microscopic physics.**

**1 mole of any substance contains the same number ($N_A$) of atoms (molecules)**

$$N_{\mathrm{A}} = \frac{\text{Mass of 1 mole of the substance}}{\text{Mass of an atom}} = 6.02214078(18) \times 10^{23}\ \mathrm{mol}^{-1}$$

**1. The Faraday constant and elementary charge**

$$F = N_{\mathrm{A}} e$$

**2. Gas constant and Boltzmann constant**

$$R = k_{\mathrm{B}} N_{\mathrm{A}}$$

**3. Molar volume and atomic volume**

$$V_{\mathrm{m}} = V_{\mathrm{atom}} N_{\mathrm{A}}$$

# Avogadro's Number measurements

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The Faraday's constant

$$F = N_{\mathrm{A}} \cdot e = 96,485.3383(83)\ \mathrm{C/mol}$$

is the electric charge transported to the electrode in an electrolytic cell, when 1 mol of singly charged ions with mass $m_X$ and elementary charge e has been deposited at the electrode.

Therefore, weighing the mass increase Δm of the electrode after a charge Q has been transferred, yields:

$$\Delta m = \frac{Q}{e} m_{\mathrm{X}} = \frac{Q}{e} \frac{M_{\mathrm{X}}}{N_{\mathrm{A}}}$$

$$\Rightarrow N_{\mathrm{A}} = \frac{Q}{e} \frac{M_{\mathrm{X}}}{\Delta m}$$

# Avogadro's Number measurements

From measurements of the absolute mass m of atoms X and the molar mass $M_X$, the Avogadro constant

$$N_A = M_X / m_X$$

can be directly determined.

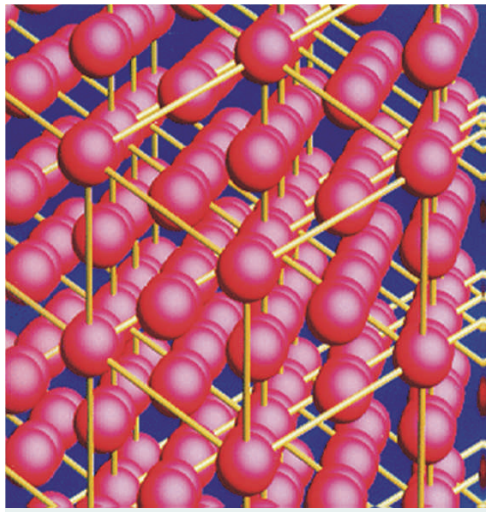
The molar mass for gas is defined as the mass of a gas of atoms X within the molar volume V = 22.4 $dm^3$ under normal conditions p and T .

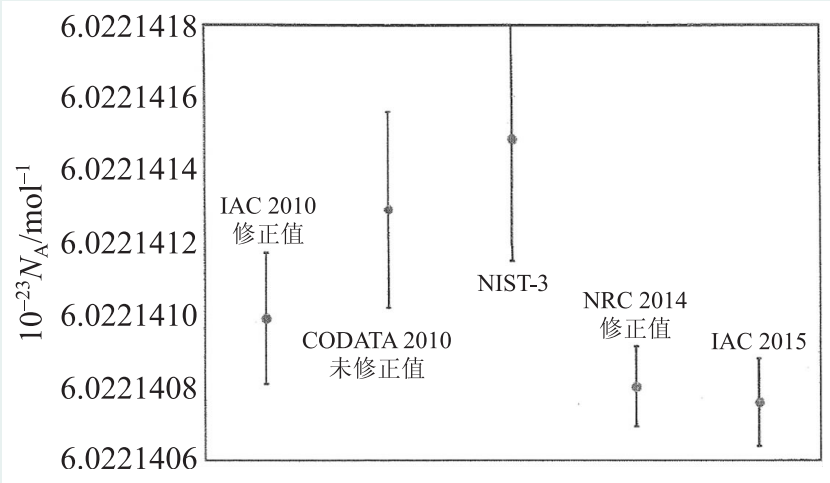
The molar mass can be also obtained for nongaseous substances from the definition

$$M_X = 0.012 m_X / m(^{12}C)\,\mathrm{kg}$$

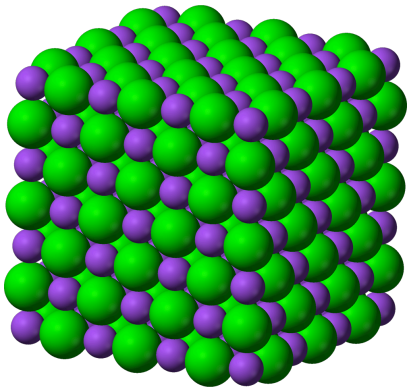
# Avogadro's Number measurements

| Method | Fundamental constant | Avogadro's number |
|---|---|---|
| General gas equation | Universal gas constant $R$ | $N_A = R/k$ |
| Barometric pressure formula (*Perrin*) | Boltzmann's constant $k$ | |
| Diffusion (*Einstein*) | | |
| Torsionsal oscillations (*Kappler*) | | |
| Electrolysis | Faraday's constant $F$ | $N_A = F/e$ |
| Millikan's oil-drop experiment | Elementary charge $e$ | |
| X-ray diffraction and interferometry | Distance $d$ between crystal planes in a cubic crystal | $N_A = (V/a^3)\frac{M_m}{M_c}$ for cubic primitive crystal |
| Measurement of atom number $N$ in a single crystal with mass $M_c$ and molar mass $M_m$ | $N_A = N \cdot \frac{M_m}{M_c}$ | $N_A = 4M_m/\varrho a^3$ for cubic face centered crystal |





# The size of atom



Assume that the masses of 1 mole atoms is A, and the atom is spherical

$$\frac{4}{3}\pi r^3 N_{\mathrm{A}} = \frac{A}{\rho}$$

The radius of atom

The density of substance

The radius of atom

$$r = \left(\frac{3A}{4\pi\rho N_{\mathrm{A}}}\right)^{\frac{1}{3}}$$

The units for the radius of atom

$$1\ \mathrm{nm} = 10^{-9}\ \mathrm{m}, \quad 1\ \text{Å} = 10^{-10}\ \mathrm{m},$$

$$1\ \mathrm{pm} = 10^{-12}\ \mathrm{m}, \quad 1\ \mathrm{fm} = 10^{-15}\ \mathrm{m}$$

# The size of atom

| Elements | A | Density ρ (g/cm³) | Radius *r* (nm) |
|---|---|---|---|
| Li | 7 | 0.7 | |
| Al | 27 | 2.7 | |
| Cu | 63 | 8.9 | |
| S | 32 | 2.07 | |
| Pb | 207 | 11.34 | |

# The size of atom

| Elements | A | Density ρ (g/cm³) | Radius *r* (nm) |
|---|---|---|---|
| Li | 7 | 0.7 | 0.16 |
| Al | 27 | 2.7 | 0.16 |
| Cu | 63 | 8.9 | 0.14 |
| S | 32 | 2.07 | 0.18 |
| Pb | 207 | 11.34 | 0.19 |

# The size of atom

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## The unit is pm

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | H 25 | | | | | | | | | | | | | | | | He |
| 2 | Li 145 | Be 105 | | | | | | | | | | | B 85 | C 70 | N 65 | O 60 | F 50 | Ne |
| 3 | Na 180 | Mg 150 | | | | | | | | | | | Al 125 | Si 110 | P 100 | S 100 | Cl 100 | Ar |
| 4 | K 220 | Ca 180 | Sc 160 | Ti 140 | V 135 | Cr 140 | Mn 140 | Fe 140 | Co 135 | Ni 135 | Cu 135 | Zn 135 | Ga 130 | Ge 125 | As 115 | Se 115 | Br 115 | Kr |
| 5 | Rb 235 | Sr 200 | Y 180 | Zr 155 | Nb 145 | Mo 145 | Tc 135 | Ru 130 | Rh 135 | Pd 140 | Ag 160 | Cd 155 | In 155 | Sn 145 | Sb 145 | Te 140 | I 140 | Xe |
| 6 | Cs 260 | Ba 215 | * | Hf 155 | Ta 145 | W 135 | Re 135 | Os 130 | Ir 135 | Pt 135 | Au 135 | Hg 150 | Tl 190 | Pb 180 | Bi 160 | Po 190 | At | Rn |
| 7 | Fr | Ra 215 | ** | Rf | Db | Sg | Bh | Hs | Mt | Ds | Rg | Cn | Nh | Fl | Mc | Lv | Ts | Og |
| | | | | | | | | | | | | | | | | | | |
| Lanthanides | * | La 195 | Ce 185 | Pr 185 | Nd 185 | Pm 185 | Sm 185 | Eu 185 | Gd 180 | Tb 175 | Dy 175 | Ho 175 | Er 175 | Tm 175 | Yb 175 | Lu 175 | | |
| Actinides | ** | Ac 195 | Th 180 | Pa 180 | U 175 | Np 175 | Pu 175 | Am 175 | Cm | Bk | Cf | Es | Fm | Md | No | Lr | | |

# Determination of the size of atom

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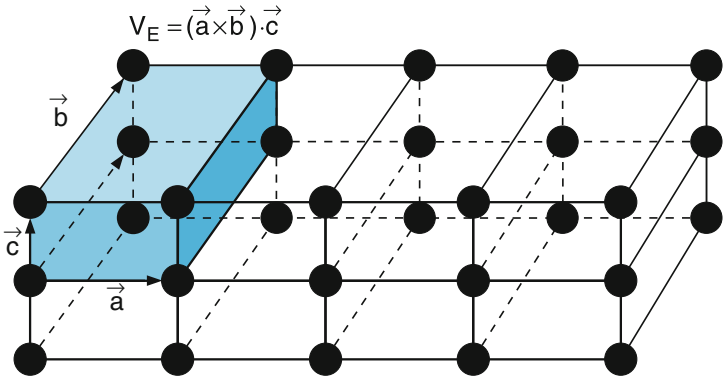
## 1. From the Covolume(协体积) in Van der Waals equation

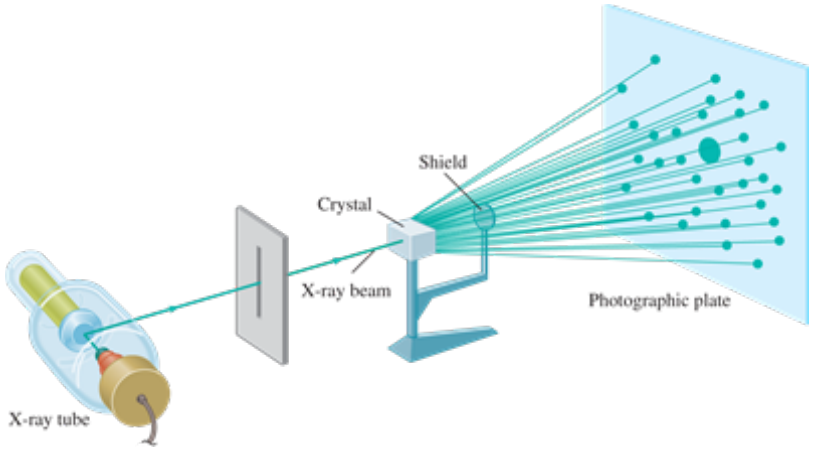
$$(P + a/V^2)(V - b) = RT$$

where, the quantity b, is equal to the fourfold volume of the particles

$$b = 4\frac{4\pi}{3}r^3 N_{\mathrm{A}}$$

## 2. From X-ray diffraction measurements on crystals

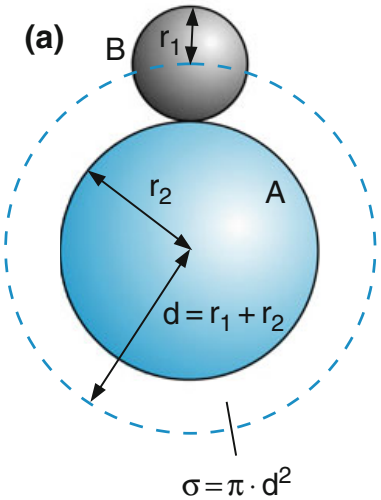


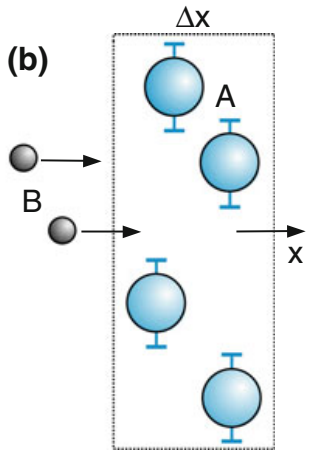



# Determination of the size of atom

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## 3. From the interaction cross section

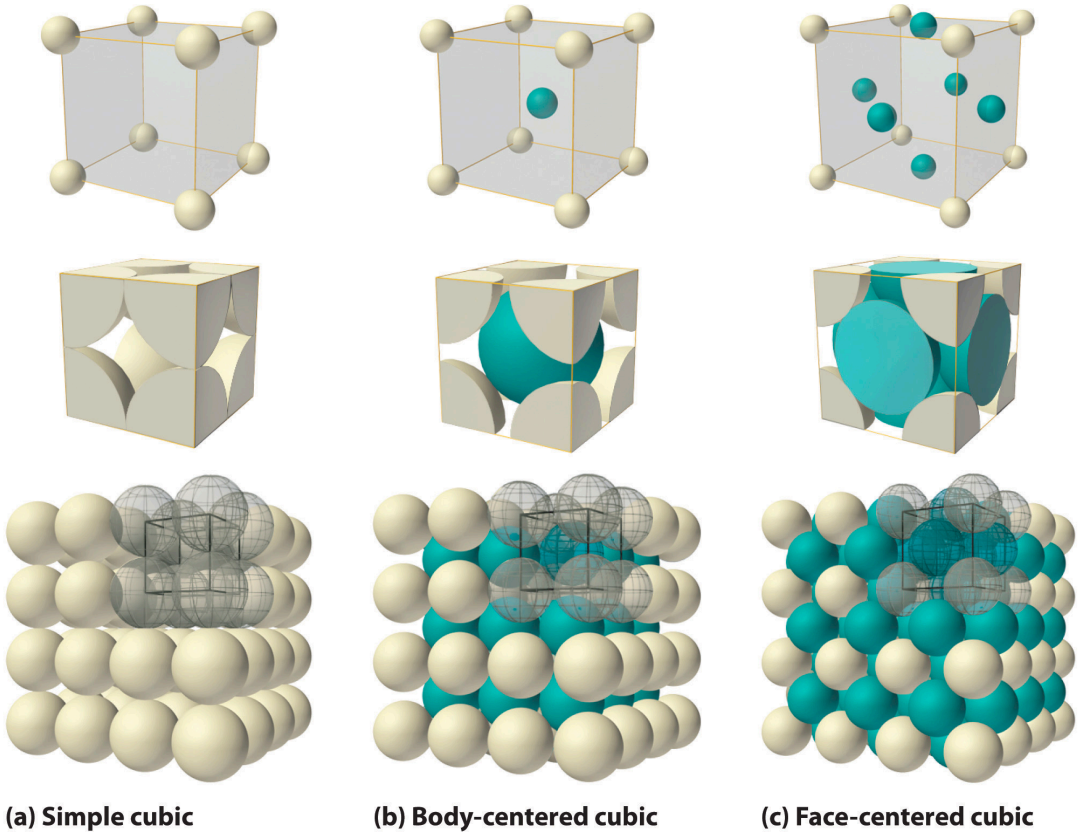



Collision probability

$P = n \cdot \sigma \cdot \Delta x$



(a) Simple cubic (b) Body-centered cubic (c) Face-centered cubic

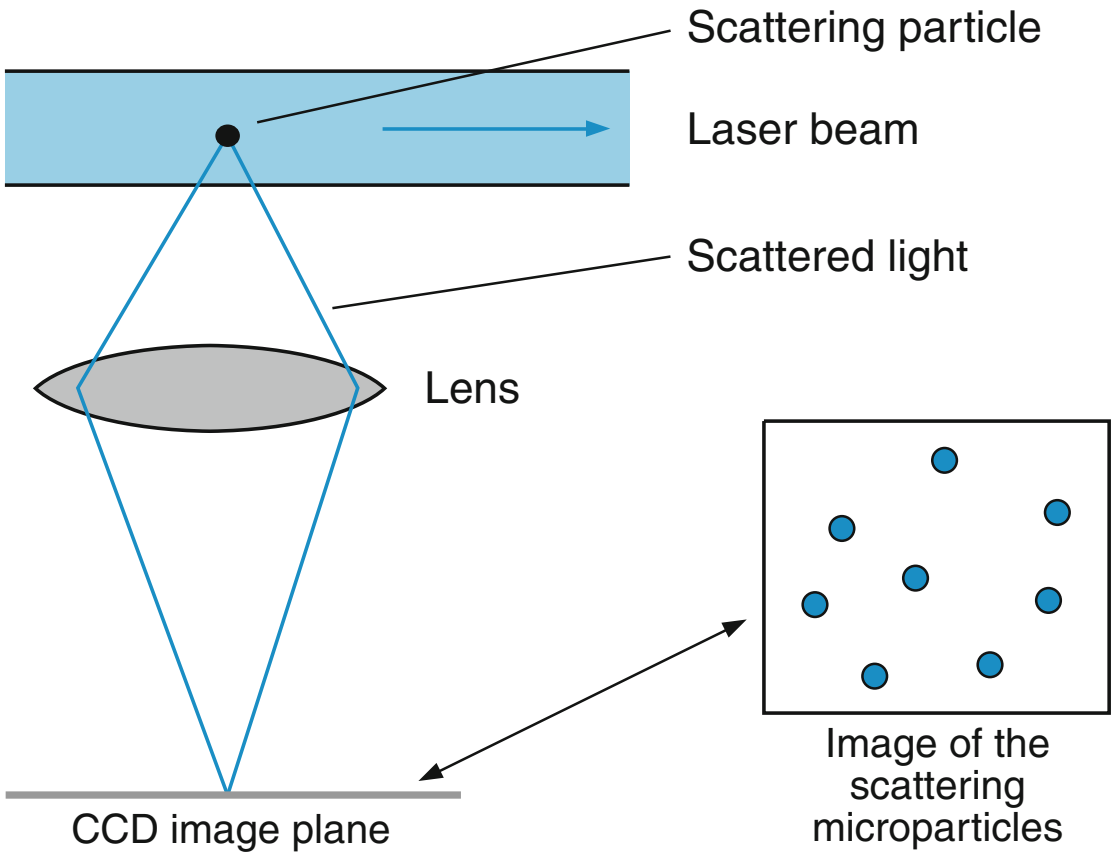
# Spallation Neutron Source



# Can One See Atoms?

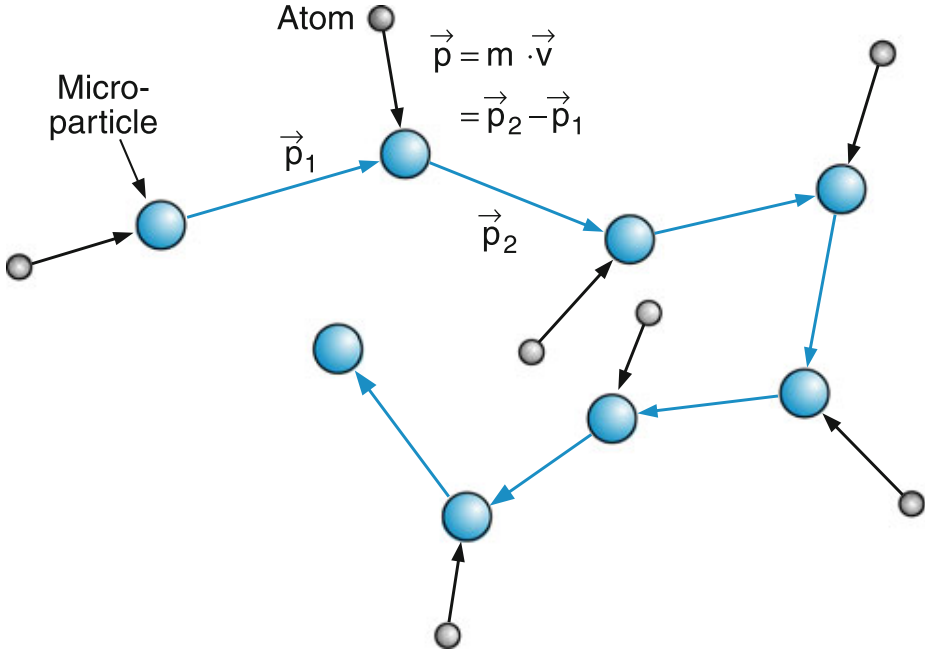
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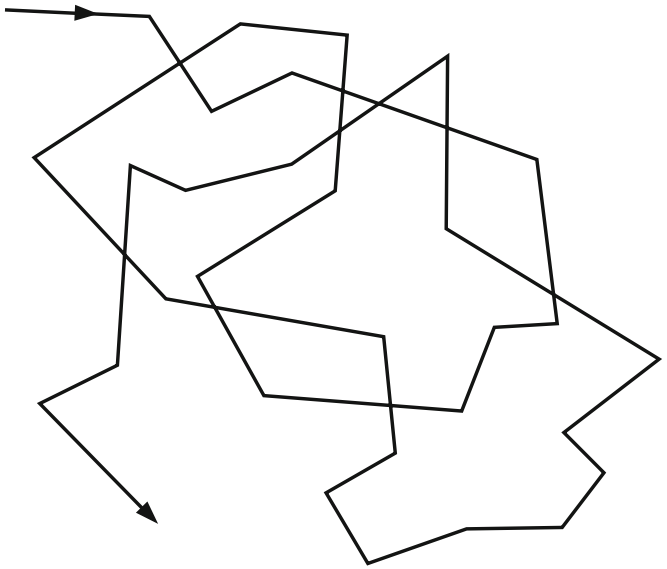
**Scattering of visible light by single atoms. Each image point corresponds to one atom**



# Can One See Atoms?

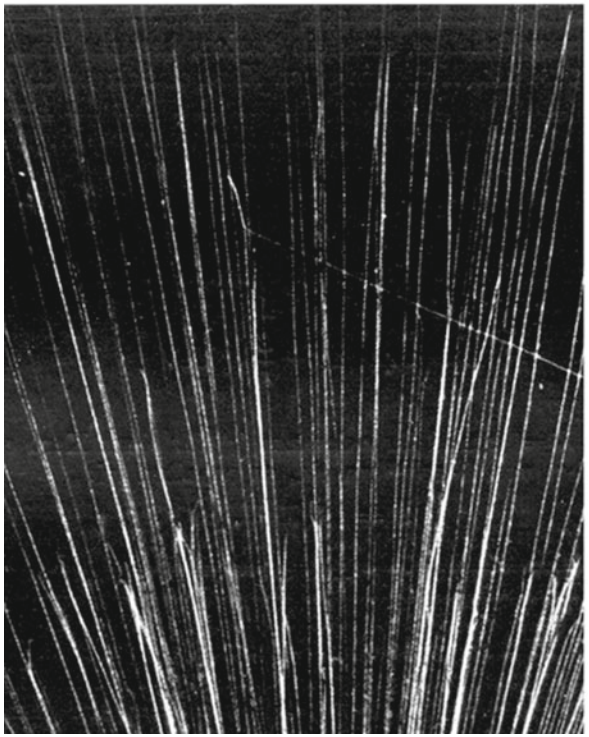
**Brownian Motion: small particles suspended in liquids performed small irregular movements, which can be viewed under a microscope**

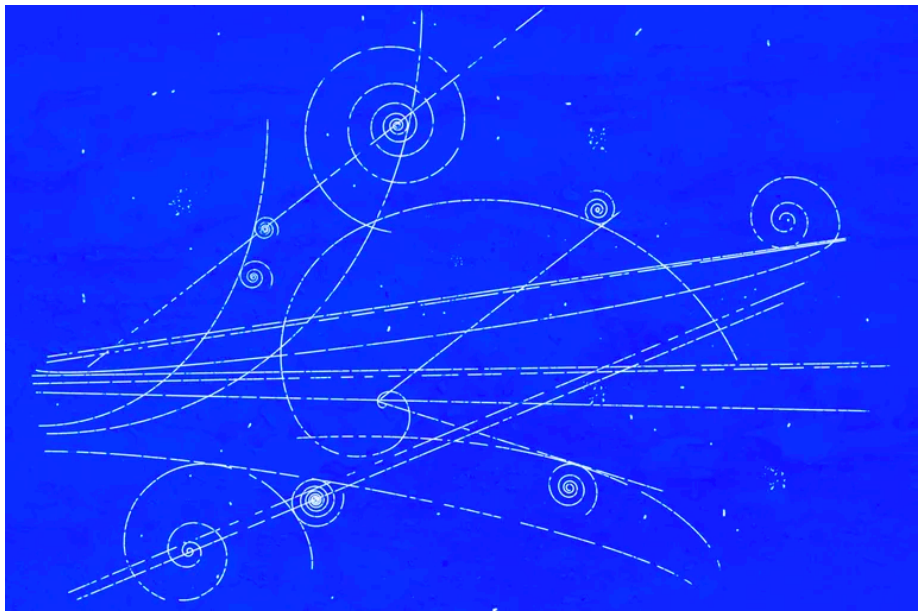


# Can One See Atoms?

Cloud Chamber: Incident particles with sufficient kinetic energy can ionize the atoms or molecules in the cloud chamber, which is filled with supersaturated water vapor.



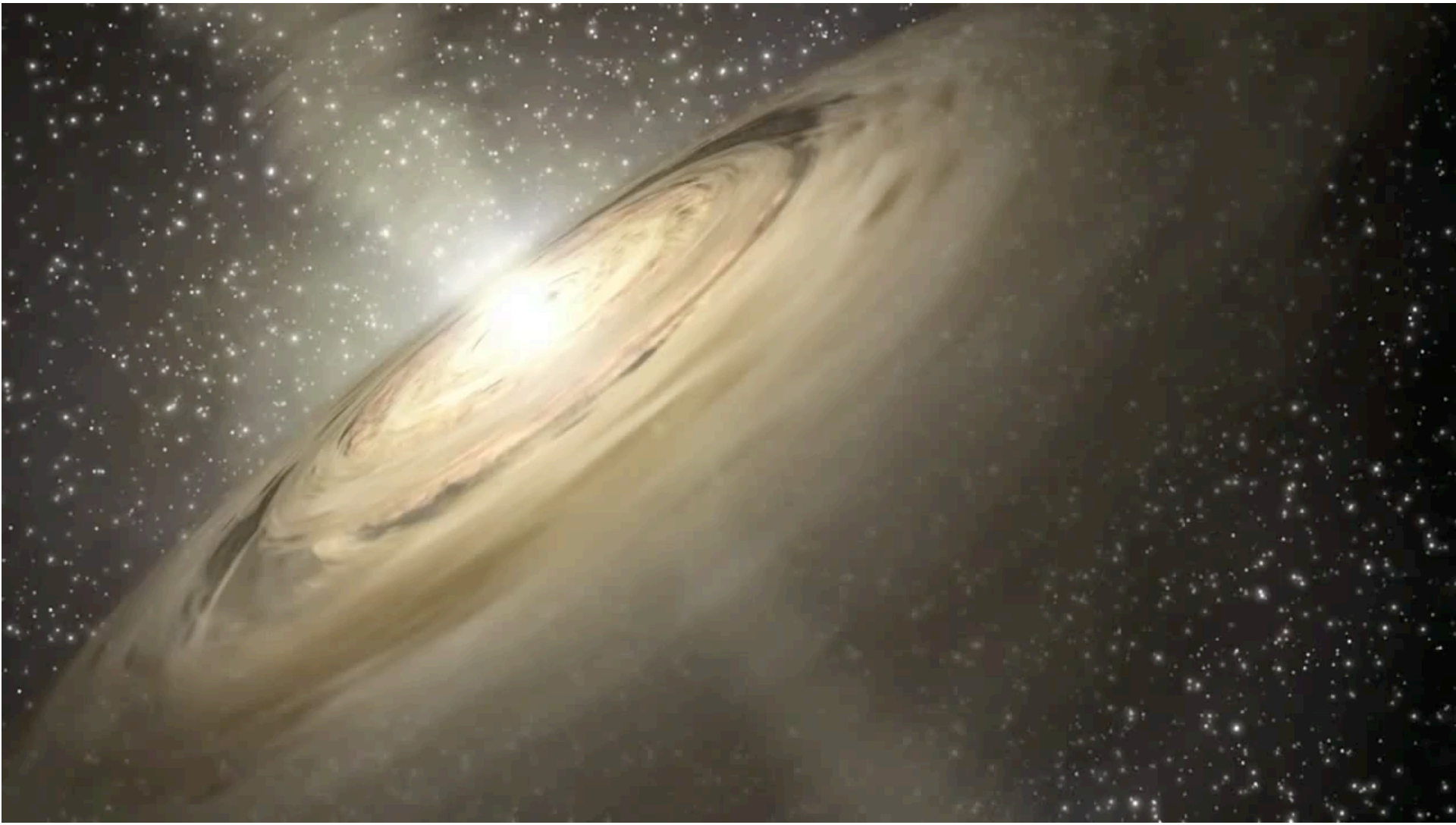


**How to Make a Cloud Chamber**

https://www.thoughtco.com/how-to-make-a-cloud-chamber-415380

# How to make a cloud chamber?

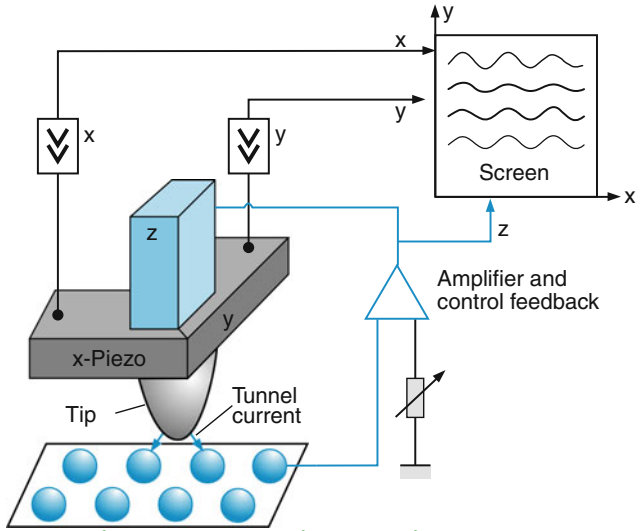
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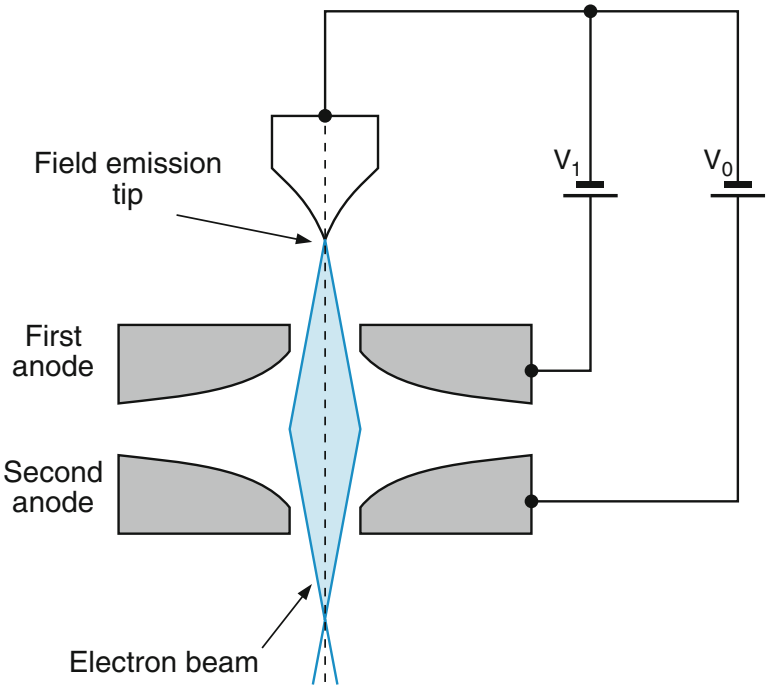
# Can One See Atoms?

## Microscopes with Atomic Resolution:

- Field Emission Microscope
- Transmission Electron Microscope
- Scanning Electron Microscope
- Scanning Tunneling Microscope
- Atomic Force Microscope

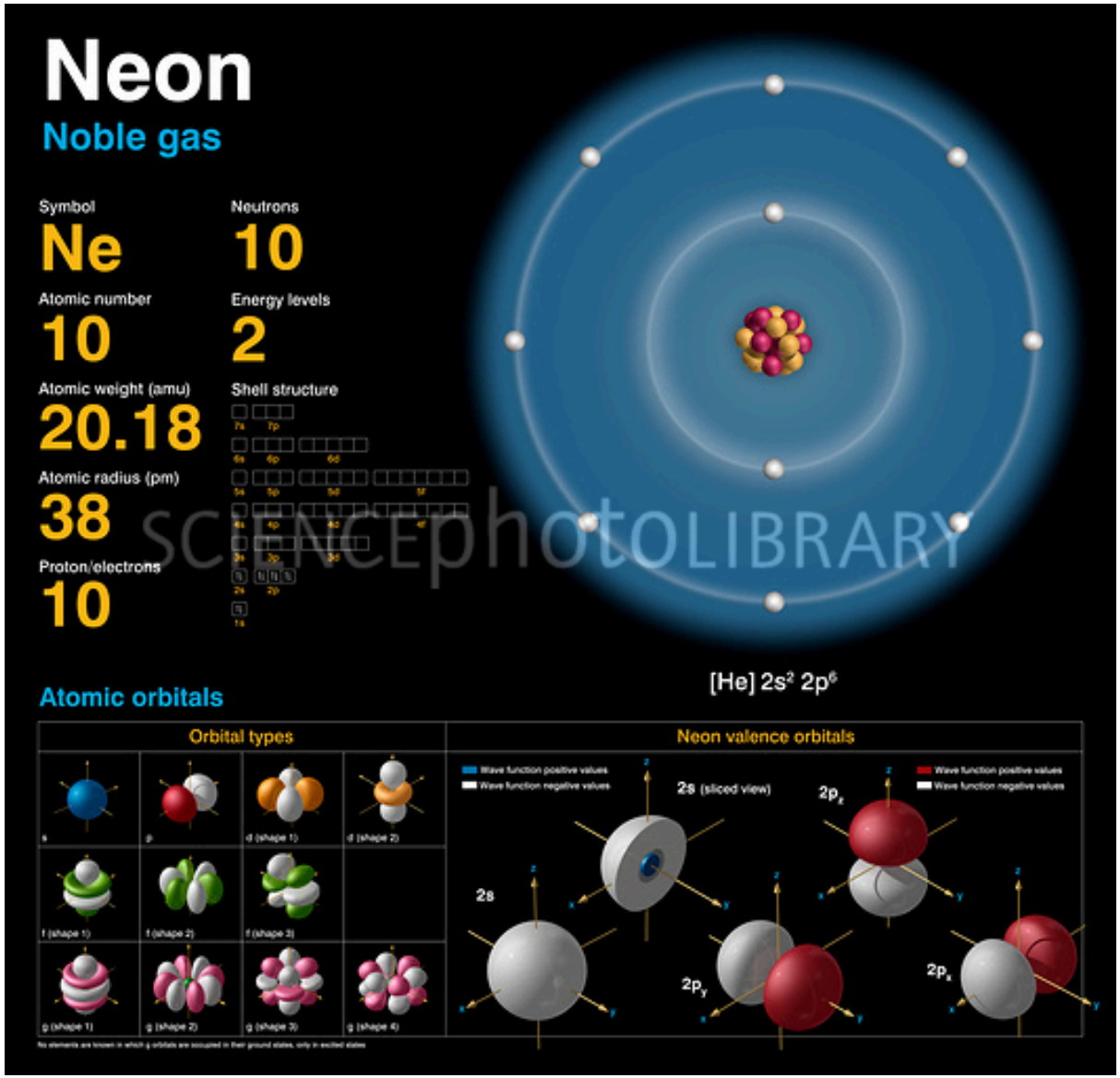



Scanning Tunneling Microscope




Field Emission Microscope

# Element symbol



# The Electric Structure of Atoms

Various experimental investigations had shown already at the end of the 19th century that matter consists of electrically charged particles. The essential evidence came from:

1. Investigations of electrolytic conductivity in polar liquids

2. Experiments on gas discharges

3. Observations of the influence of magnetic fields on the electric current in metals and semiconductors

4. The discovery that particles emitted from radioactive substances show different deflections in magnetic fields.

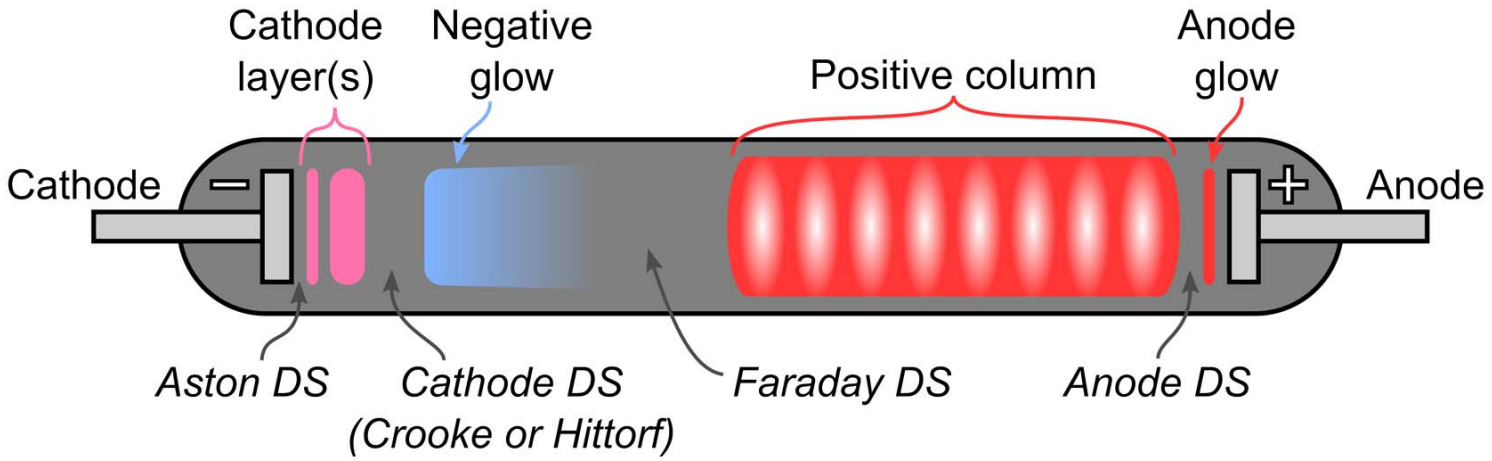
# The Electric Structure of Atoms

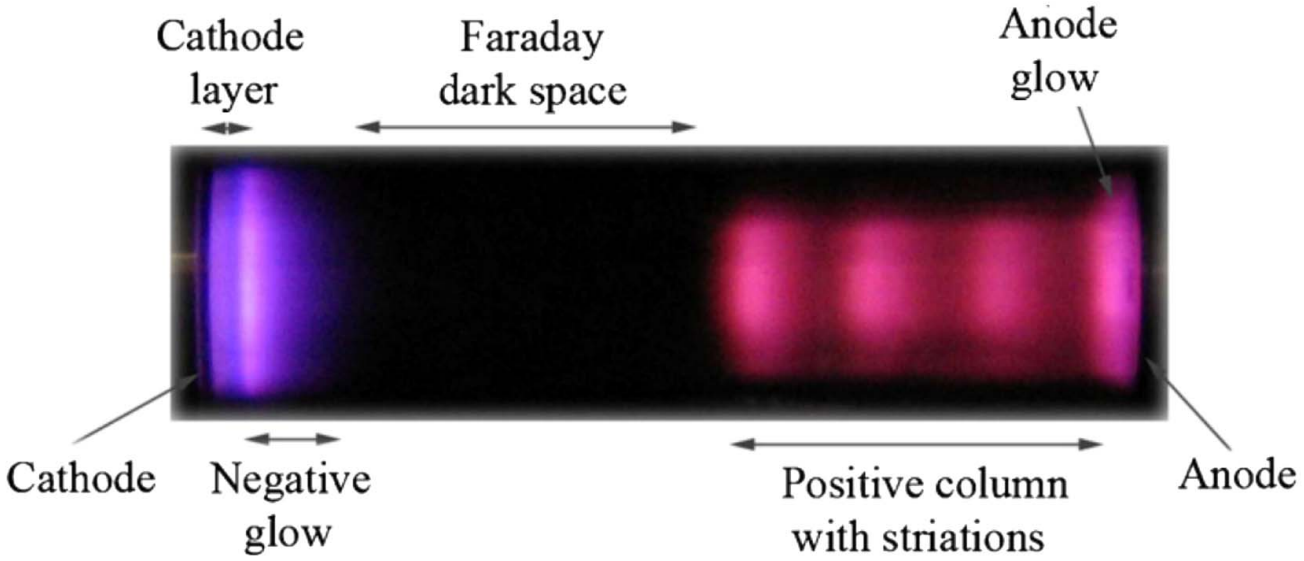
Atoms are built up of charged particles. They can therefore not be "indivisible," but have a substructure, which, however, was unknown at this time. The electrically charged positive and negative constituents have different masses.

This raises the questions:

- What properties do these constituents have?
- What force keeps them together to form stable atoms?
- What is the charge distribution inside the atom?
- How can the microscopic properties of matter be explained by this model?

# Faraday dark space



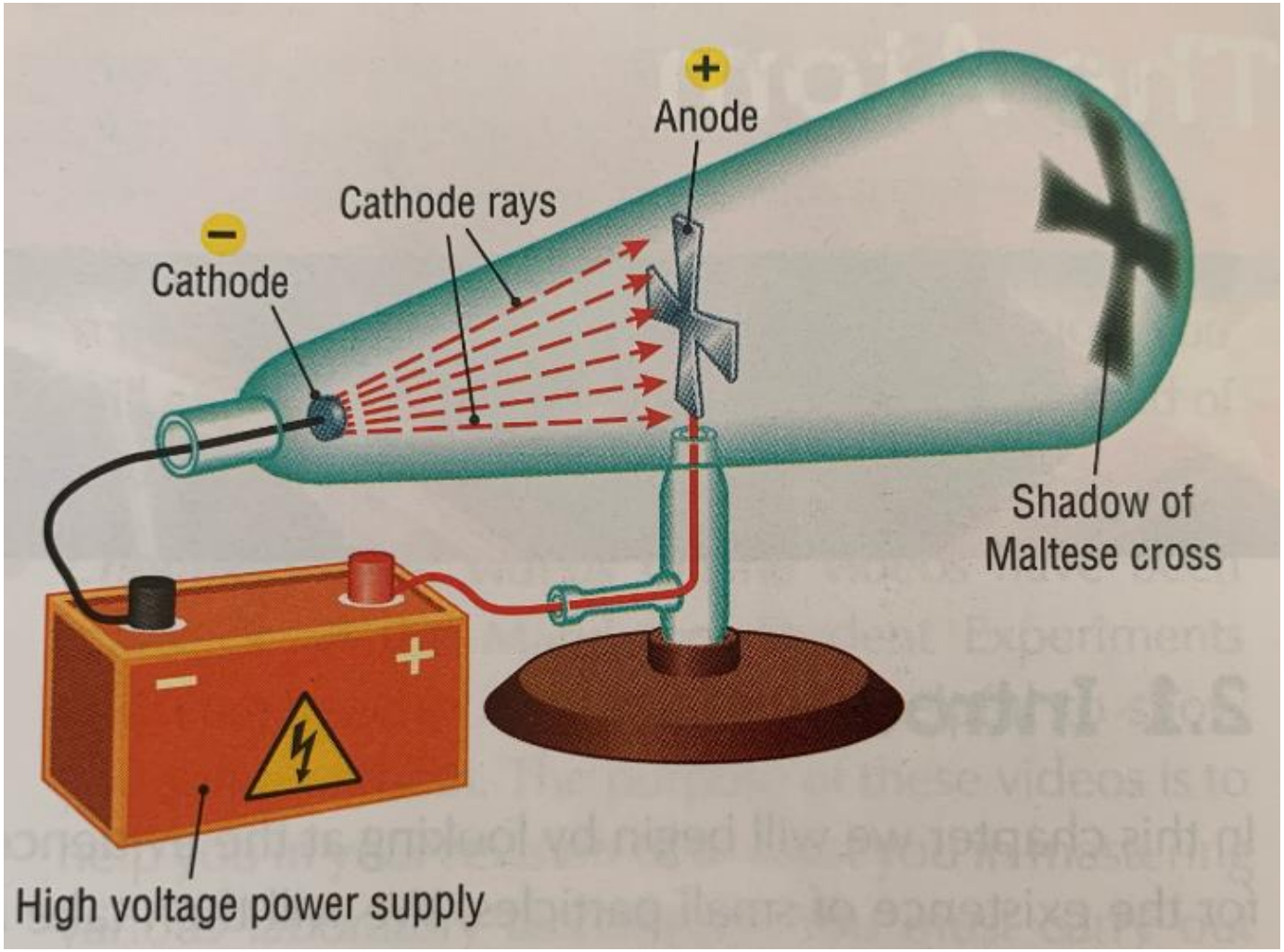



# The Electric Structure of Atoms

Investigations of gas discharges have all contributed much to our understanding of the electric structure of atoms. It is worthwhile to note that the essential experimental progress was only possible after the improvement of vacuum technology.

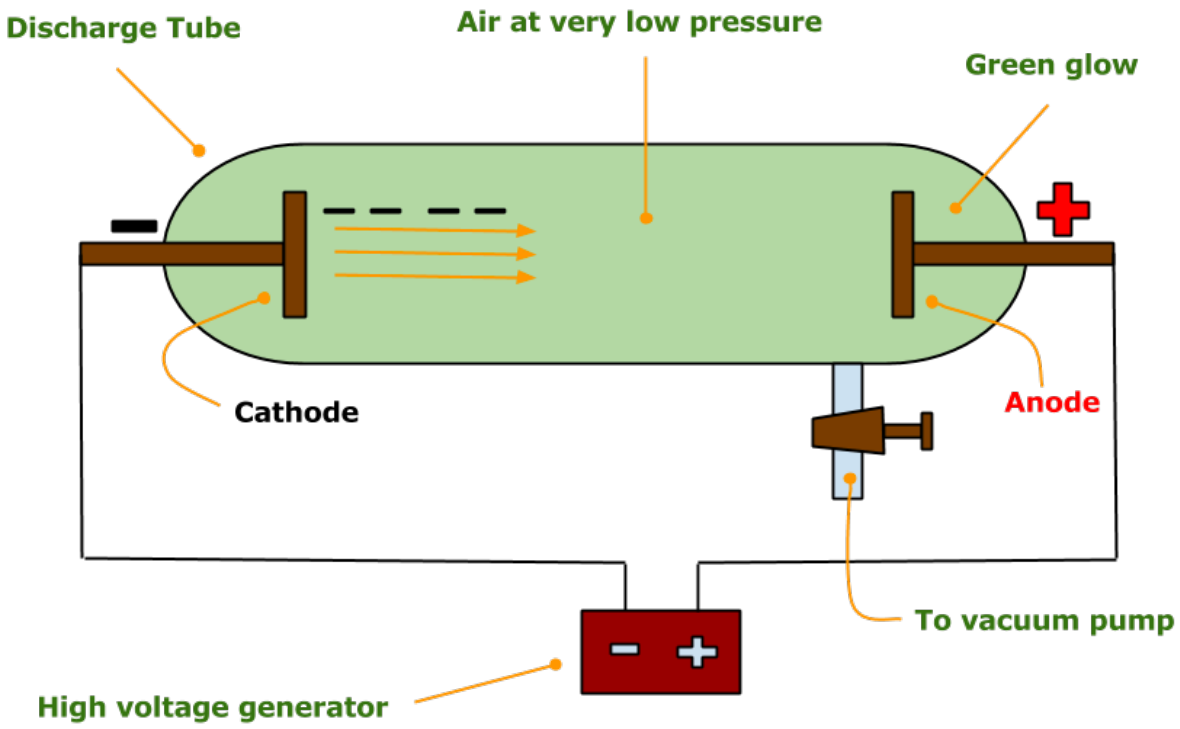
In a gas discharge tube at low pressures, Hittorf observed particle rays emitted from the cathode that followed (without external fields) straight lines. He could prove this by the shadow that was produced on a fluorescent screen when obstacles were put in the path of the cathode rays.

# The Crookes Tube



# The discovery of electron

## Cathode ray tube



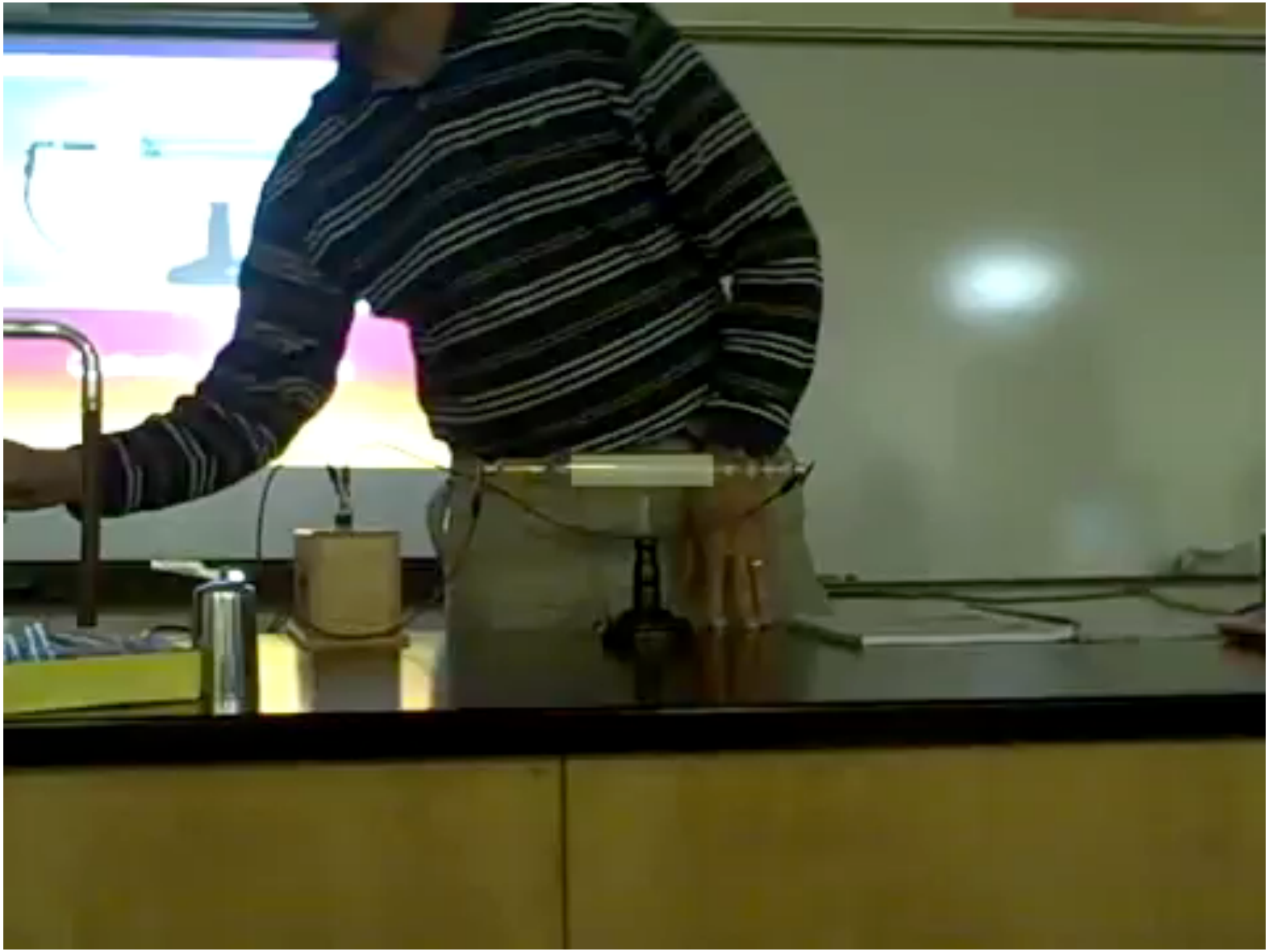

Production of cathode rays

Obtained in 1895 by J.B. Perrin and with an improved apparatus in 1897 by Thomson

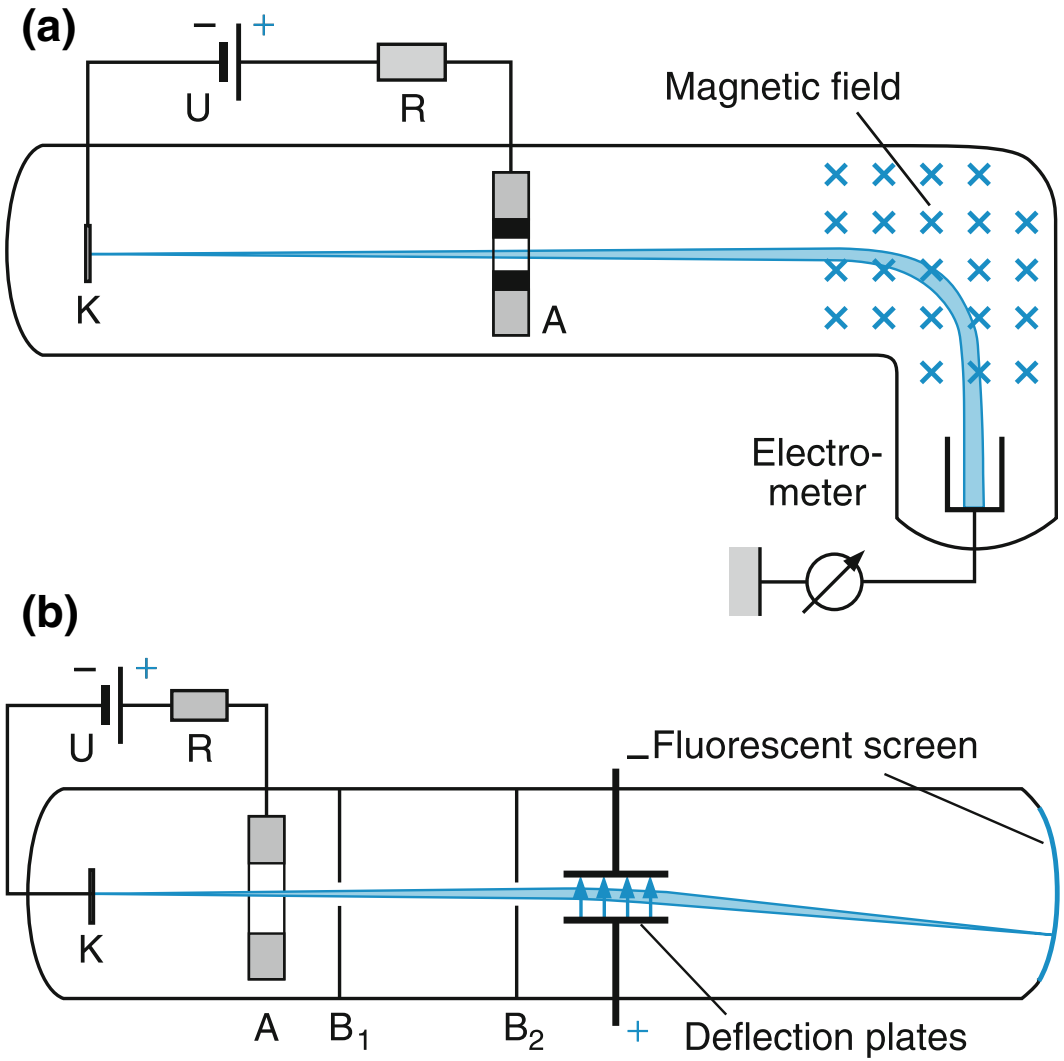




# Cathode ray tube

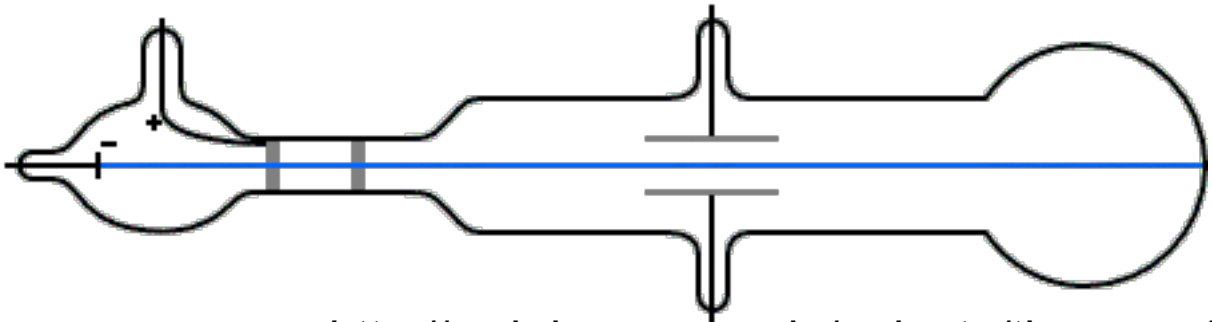


# Cathode ray in external field



# The discovery of electron

## 1897, J. J. Thomson found electron (corpuscles)



http://web.lemoyne.edu/~giunta/thomson1897.html

1. They travel in straight lines.
2. They are independent of the material composition of the cathode.
3. Applying electric field in the path of cathode ray deflects the ray towards positively charged plate. Hence cathode ray consists of negatively charged particles.

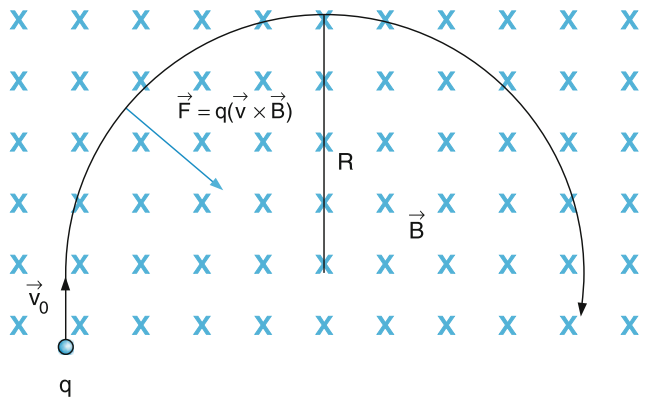
# The discovery of electron

The negative light particles of the cathode rays were named electrons after a proposal by J. Stoney and G. Fitzgerald in 1897. The positively charged heavy particles were named ions according to the existing name for charged atoms or molecules in the electrolysis.

These discoveries gave the following picture of the charged constituents of atoms: atoms consist of negatively charged electrons and positively charged particles that just compensate the negative charge to make the whole atom neutral.

# The discovery of electron

## Charge-to-Mass Ratio for the Electron




### Lorentz Force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

### Circle motion

$$\frac{mv^2}{R} = evB_z$$

### Acceleration voltage V

$$(m/2)v^2 = eV$$

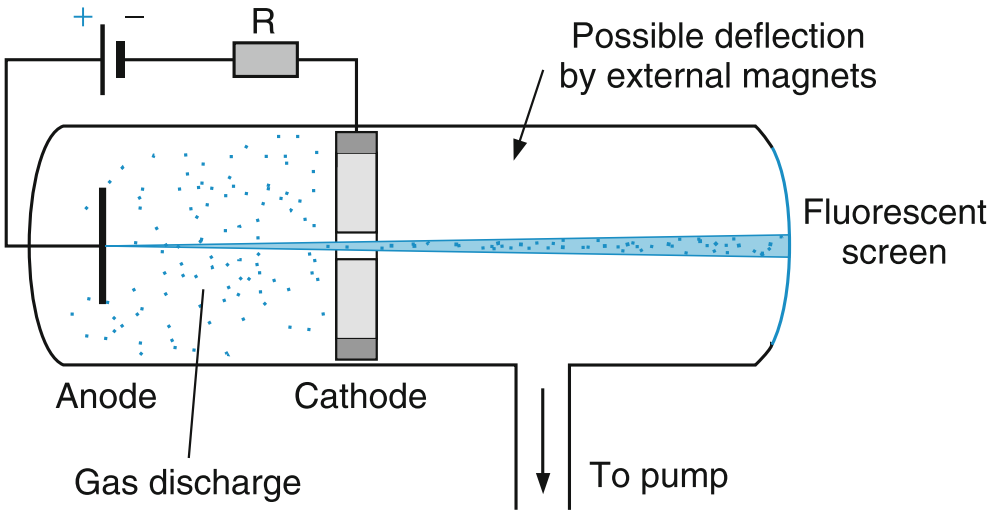
### Charge-to-Mass ration

$$\frac{e}{m} = \frac{2V}{R^2 B^2}.$$

| Gas. | Value of W/ | I. | m/e | v. |
|---|---|---|---|---|
| | | Tube | | |
| Air | $4.6\times10^{11}$ | 230 | $.57\times10^{-7}$ | $4\times10^{9}$ |
| Air | $1.8\times10^{12}$ | 350 | $.34\times10^{-7}$ | $1\times10^{10}$ |
| Air | $6.1\times10^{11}$ | 230 | $.43\times10^{-7}$ | $5.4\times10^{9}$ |
| Air | $2.5\times10^{12}$ | 400 | $.32\times10^{-7}$ | $1.2\times10^{10}$ |
| Air | $5.5\times10^{11}$ | 230 | $.48\times10^{-7}$ | $4.8\times10^{9}$ |
| Air | $1\times10^{12}$ | 285 | $.4\times10^{-7}$ | $7\times10^{9}$ |
| Air | $1\times10^{12}$ | 285 | $.4\times10^{-7}$ | $7\times10^{9}$ |
| Hydrogen | $6\times10^{12}$ | 205 | $.35\times10^{-7}$ | $6\times10^{9}$ |
| Hydrogen | $2.1\times10^{12}$ | 460 | $.5\times10^{-7}$ | $9.2\times10^{9}$ |
| Carbonic | $8.4\times10^{11}$ | 260 | $.4\times10^{-7}$ | $7.5\times10^{9}$ |
| Carbonic | $1.47\times10^{12}$ | 340 | $.4\times10^{-7}$ | $8.5\times10^{9}$ |
| Carbonic | $3.0\times10^{12}$ | 480 | $.39\times10^{-7}$ | $1.3\times10^{10}$ |

# Kanalstrahlen

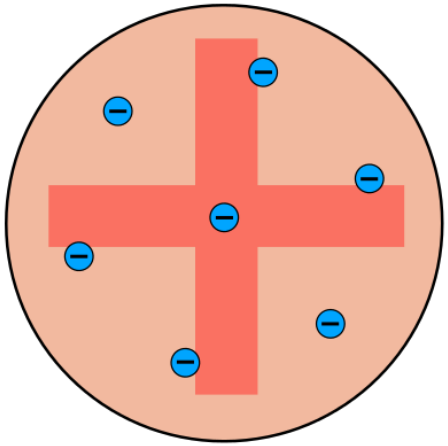
Thomson could also show that the ratio e/m was independent of the cathode material, but was about $10^4$ times larger than that for the "Kanalstrahlen" discovered in 1886 by Eugen Goldstein (1850–1930) in a discharge tube, which fly through a hole in the cathode in the opposite direction of the cathode rays.

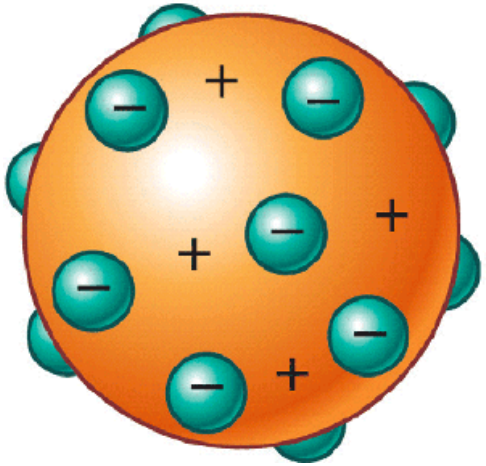



Wilhelm Wien (1864–1928) measured in 1897 the value of e/m for the particles in the Kanalstrahlen and he proved that they are positively charged atoms of the gas inside the discharge tube.

# Contemporary theories of atom

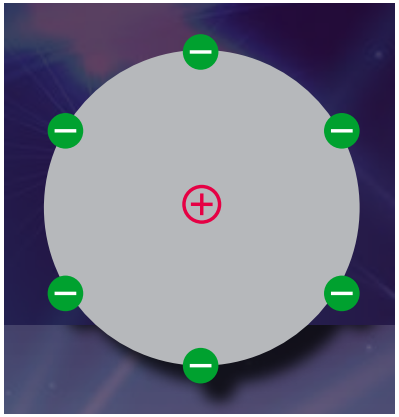
## 1. Plum pudding model (Lord Kelvin and J. J. Thomson)



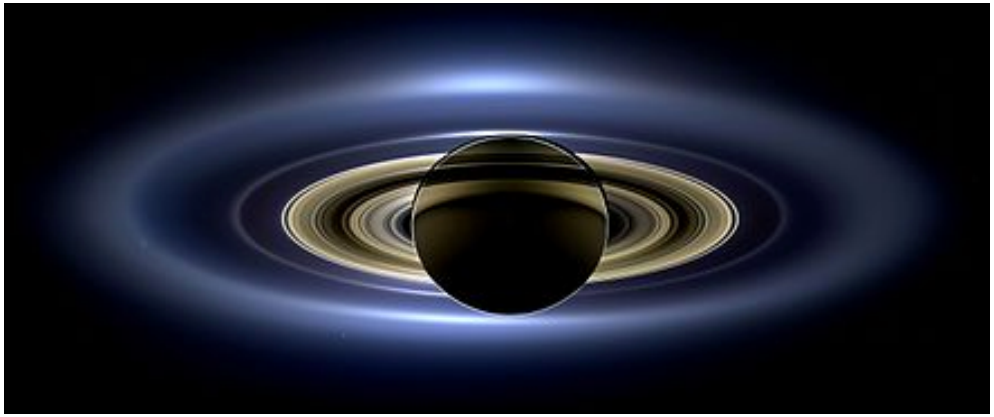





## 2. Saturnian model of the atom (H. Nagaoka)





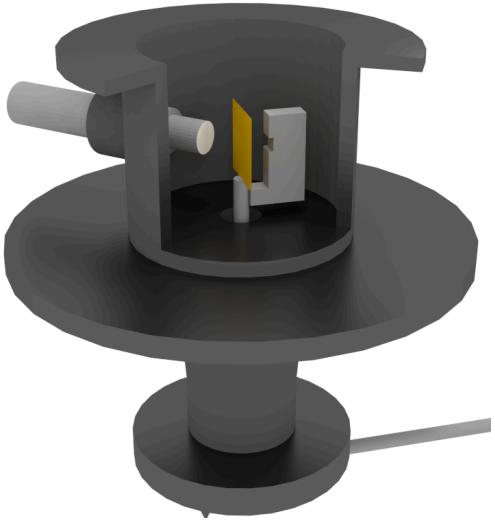
# Plum pudding model

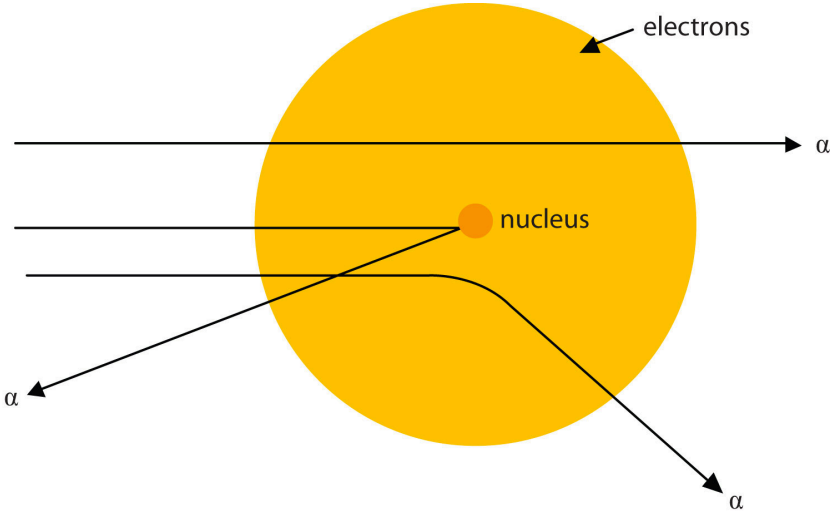
1. Each negatively-charged electron was paired with a positively-charged particle that followed it everywhere within the atom.

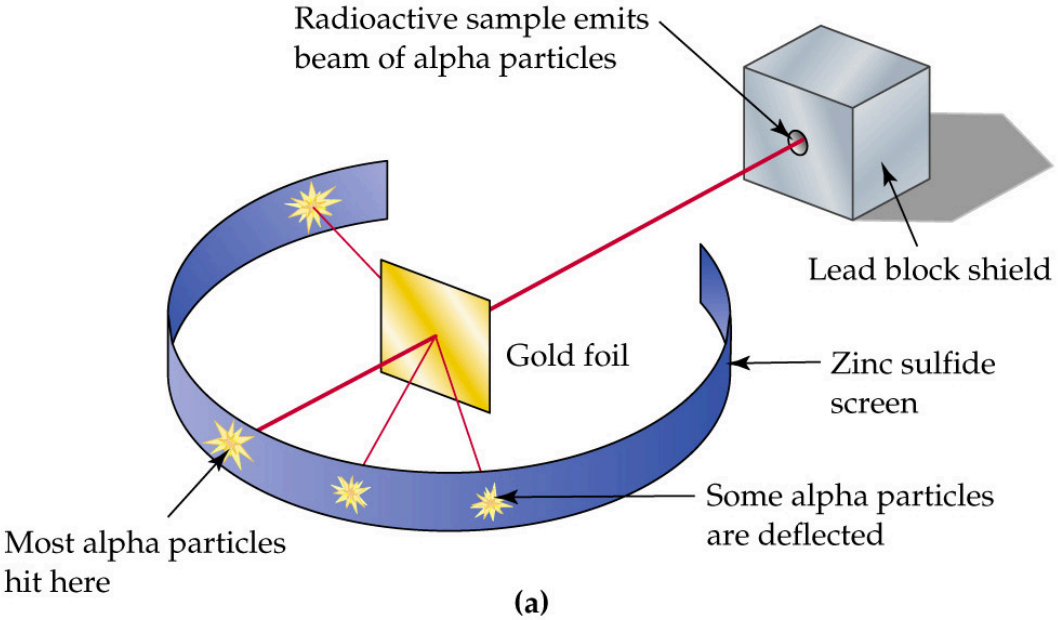
2. Negatively-charged electrons orbited a central region of positive charge having the same magnitude as the total charge of all the electrons.

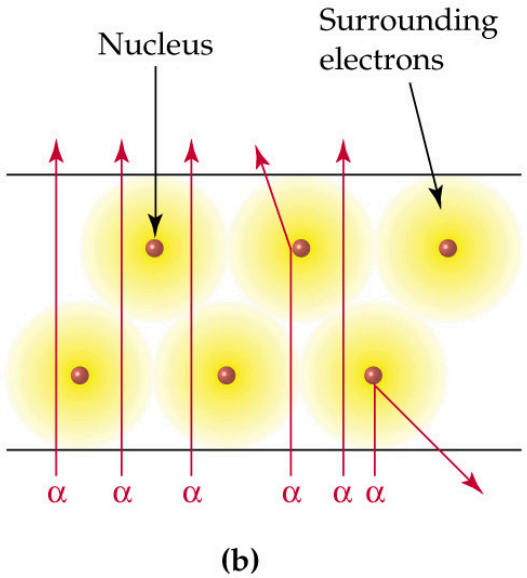
3. The negative electrons occupied a region of space that was uniformly positively charged (often considered as a kind of "soup" or "cloud" of positive charge).

# Geiger-Marsden experiment



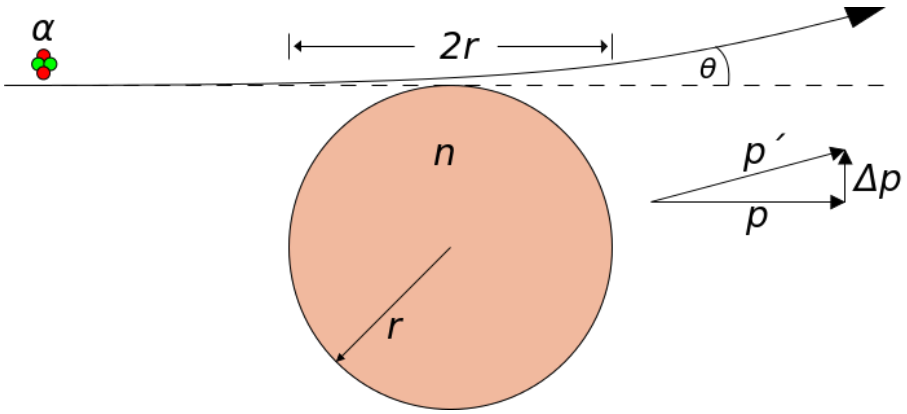


# Implications of plum pudding model

Using classical physics, the alpha particle's lateral change in momentum $\Delta p$ can be approximated using the impulse of force relationship and the Coulomb force expression:




$$\Delta p = F\Delta t = k\frac{Q_\alpha Q_n}{r^2}\frac{2r}{v_\alpha}$$

The maximum deflection angle:

$$\theta \approx \frac{\Delta p}{p} < k\frac{2Q_\alpha Q_n}{m_\alpha r v_\alpha^2} = 0.000326 \text{ rad}$$

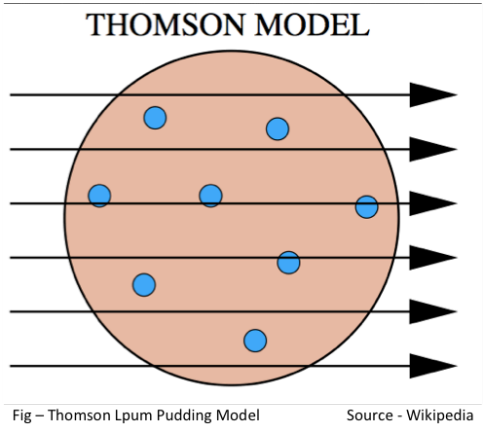



Fig – Thomson Lpum Pudding Model Source - Wikipedia

where, $r$: radius of a gold atom

$Q_n$: positive charge of gold atom

$Q_\alpha$: charge of alpha particle.

$k$: Coulomb's constant

$m_\alpha$: mass of alpha particle

$v_\alpha$: velocity of alpha particle

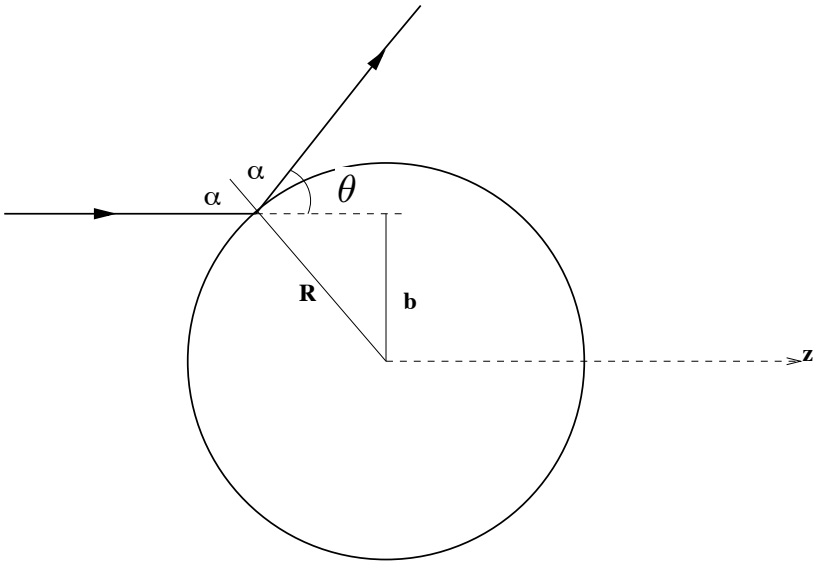
# Scattering in a hard sphere

Consider one of these particles incident with velocity v and impact parameter b.

R: radius of the sphere

α: incident angle

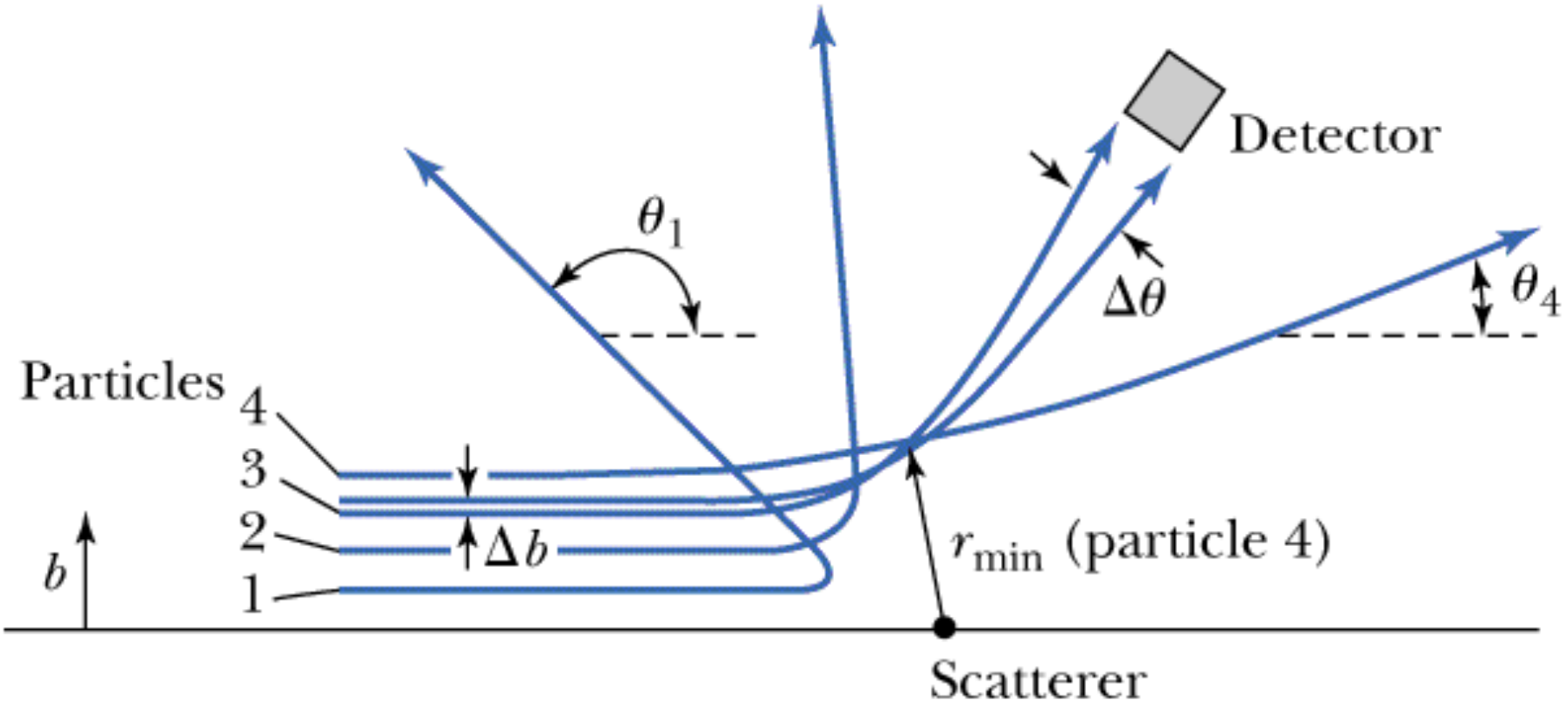
b: impact parameter $b = R\sin\alpha$




The particle is deflected through an angle $\theta = \pi - 2\alpha$, related to the impact parameter bv

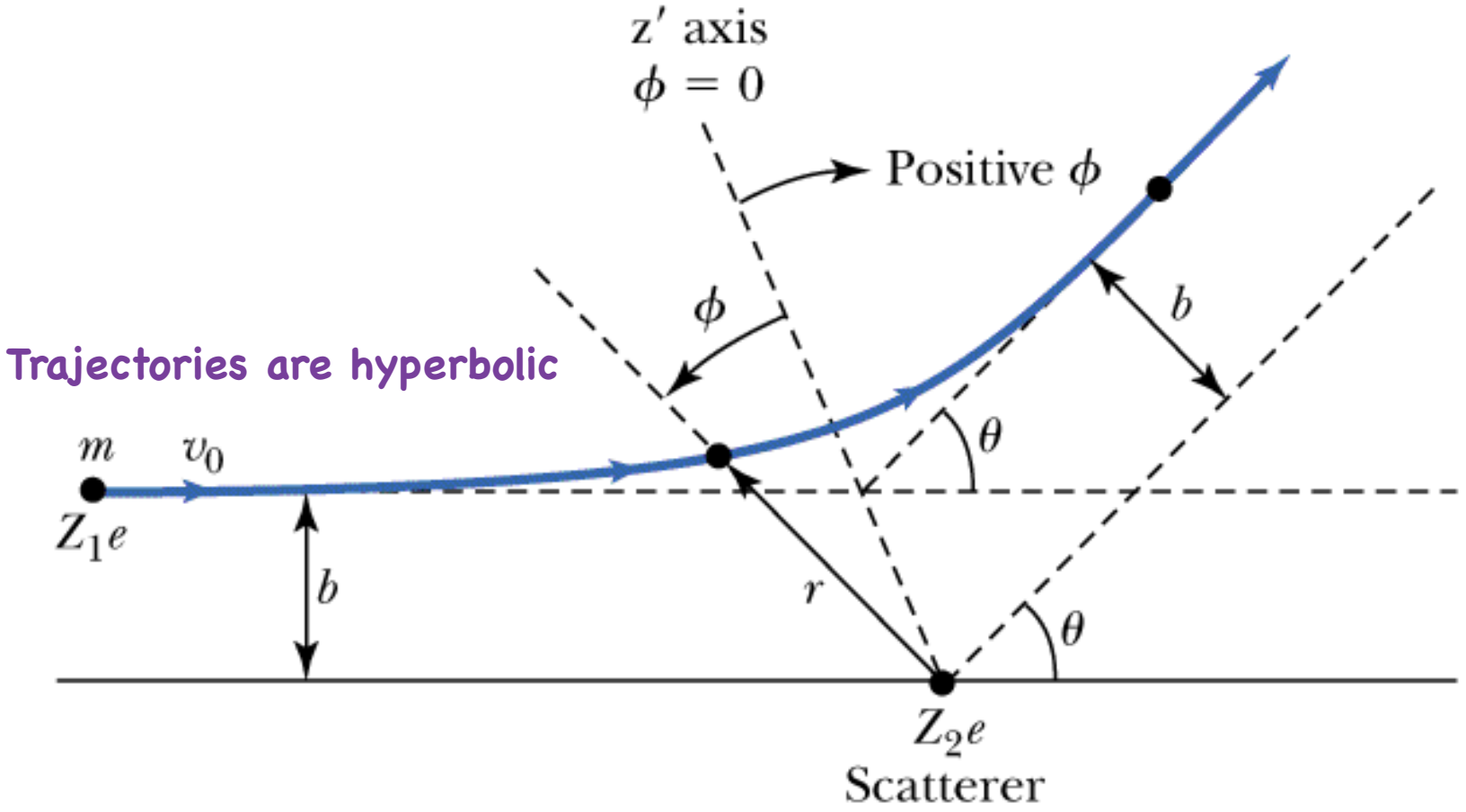
$$b = R\cos\frac{\theta}{2}.$$

Trajectories are strongly dependent on the impact parameter



# Rutherford Scattering Formula

The key concept in Rutherford scattering is the relationship between the impact parameter $b$ and the scattering angle $\vartheta$.



# Rutherford Scattering Formula

## 1. Basic knowledge

The Coulomb force;

The Newton's laws;

The conservation of linear momentum;

The conservation of angular momentum.

## 2. Assumptions

Single scattering

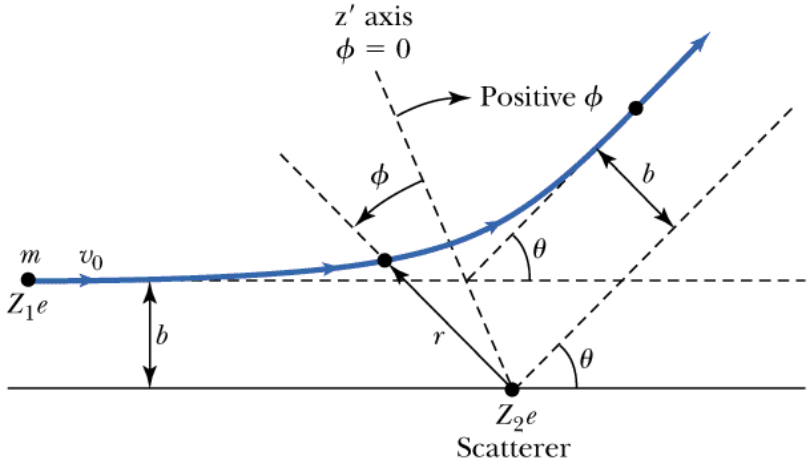
Only consideration Coulomb force

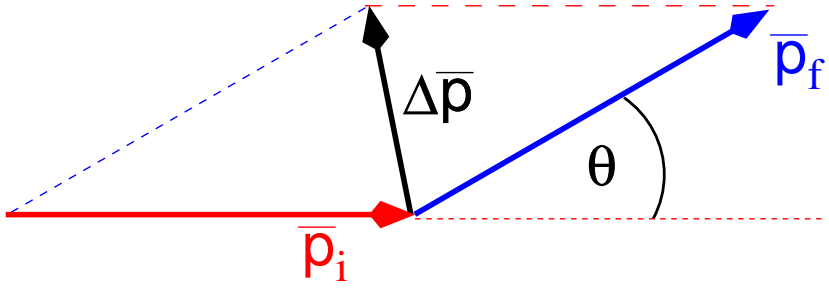
The effect of electrons in nuclei is neglected

The target is static

# Rutherford Scattering Formula

## Momentum change in Rutherford scattering



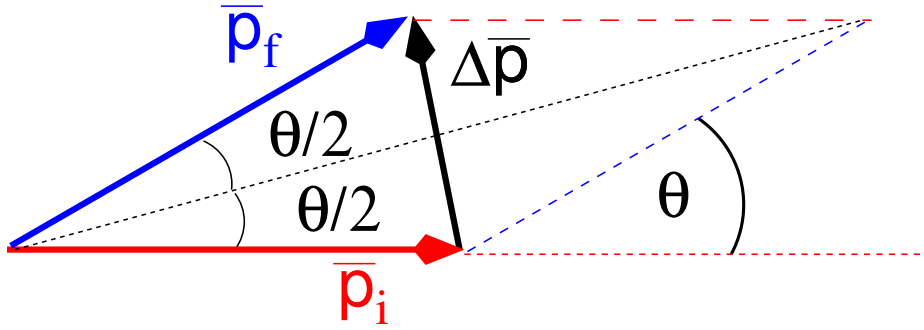




**Elastic scattering**

$$|\vec{p}_i| = |\vec{p}_f| = p$$

**Momentum change**

$$\Delta p = 2p\sin(\theta/2)$$



# Rutherford Scattering Formula

From the Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt} \Longrightarrow \Delta\vec{p} = \int \vec{F} dt$$

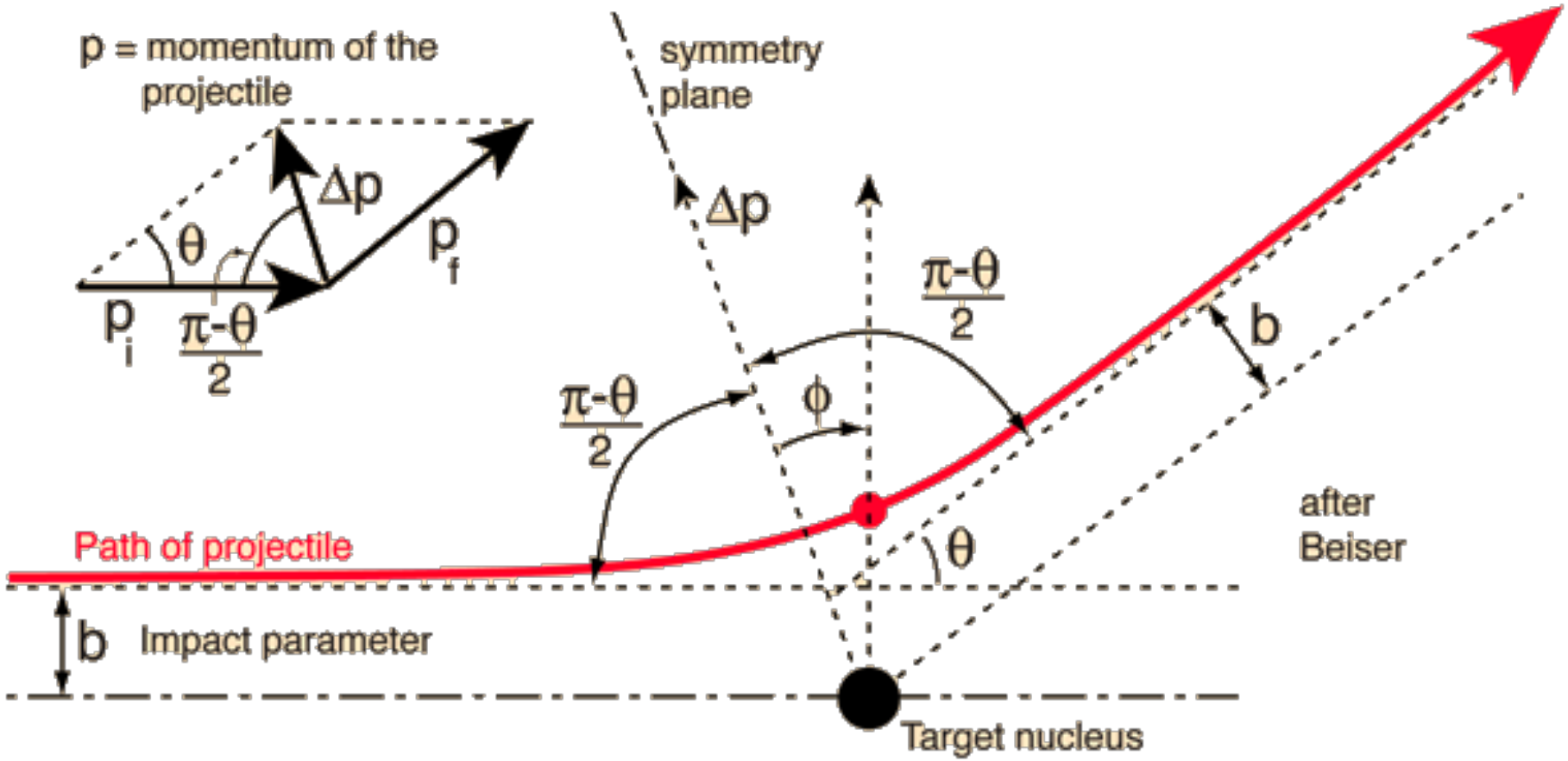
The force is the Coulomb force

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e^2}{r^2} \frac{\vec{r}}{r}$$

Before we start integrating let us note that the trajectories are symmetric with respect to the line defined by the distance of the closest approach

# Rutherford Scattering Formula

## Trajectories are symmetric with respect to angle ϕ



# Rutherford Scattering Formula

The symmetry with respect to the line at ϕ = 0 implies

$$\Delta\vec{p} = \int \vec{F}dt \Longrightarrow \Delta p = \int F\cos\phi dt$$

So,

$$\Delta p = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\int \frac{1}{r^2}\cos\phi dt$$

This integral can be carried over with a help of conservation of angular momentum.

The angular momentum is

$$\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} = m\vec{r}\times\left(\frac{d\vec{r}}{dt} + r\frac{d\vec{\phi}}{dt}\right) = mr\vec{r}\times\frac{d\vec{\phi}}{dt}$$

# Rutherford Scattering Formula

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**The magnitude of angular momentum**

$$L = |\vec{L}| = mr^2 \frac{d\vec{\phi}}{dt}$$

**From the initial condition**

$$L = mv_0 b$$

**Since the angular momentum is conserved**

$$mr^2 \frac{d\phi}{dt} = mv_0 b$$

$$\frac{dt}{r^2} = \frac{d\phi}{v_0 b}$$

Thus the change of momentum

$$\Delta p = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\int \frac{dt}{r^2}\cos\phi = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\int \frac{d\phi}{v_0 b}\cos\phi$$

$$= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{v_0 b}\int_{\phi_<}^{\phi_>} d\phi\cos\phi$$

The limits for integration are

$$\phi_> = \frac{1}{2}(\pi - \theta)$$

$$\phi_< = -\frac{1}{2}(\pi - \theta)$$

# Rutherford Scattering Formula

The integral is

$$\Delta p = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{v_0 b}\int_{\phi_<}^{\phi_>} d\phi\cos\phi = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{v_0 b}(\sin\phi_> - \sin\phi_<)$$

$$= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{2}{v_0 b}\cos\frac{\theta}{2}$$

Since

$$\Delta p = 2p\sin(\theta/2)$$

The impact parameter is expressed as

$$b = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{pv_0}\frac{1}{\tan(\theta/2)}$$

$$= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{2E}\frac{1}{\tan(\theta/2)}$$

with E being the initial kinetic energy for the projectile

## The reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

## Energy conservation demands

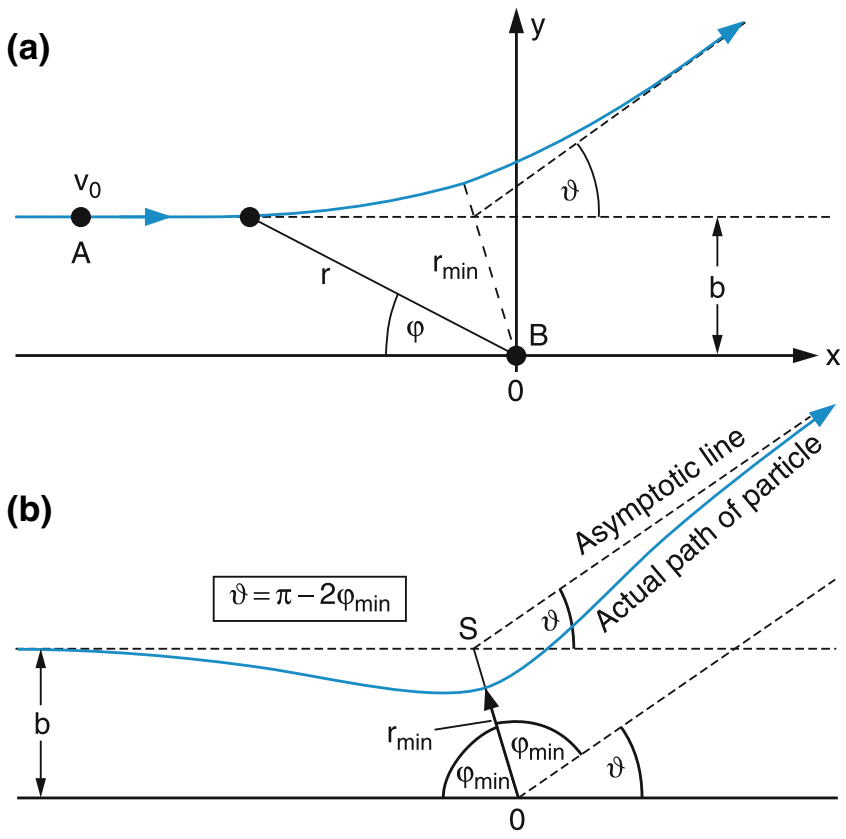
$$\frac{1}{2}\mu v^2 + E_{\text{pot}}(r) = \frac{1}{2}\mu v_0^2 = \text{const},$$

## $v_0$ is the initial velocity

## The angular momentum L

$$\boldsymbol{L} = \mu(\boldsymbol{r} \times \boldsymbol{v}) = \mu\left(\boldsymbol{r} \times \left[\frac{\mathrm{d}r}{\mathrm{d}t}\hat{e}_r + r\frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{e}_t\right]\right)$$

$$= \mu r\dot{\varphi}\left(\boldsymbol{r} \times \hat{e}_t\right),$$



## The reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

## Energy conservation demands

$$\frac{1}{2}\mu v^2 + E_{\text{pot}}(r) = \frac{1}{2}\mu v_0^2 = \text{const},$$

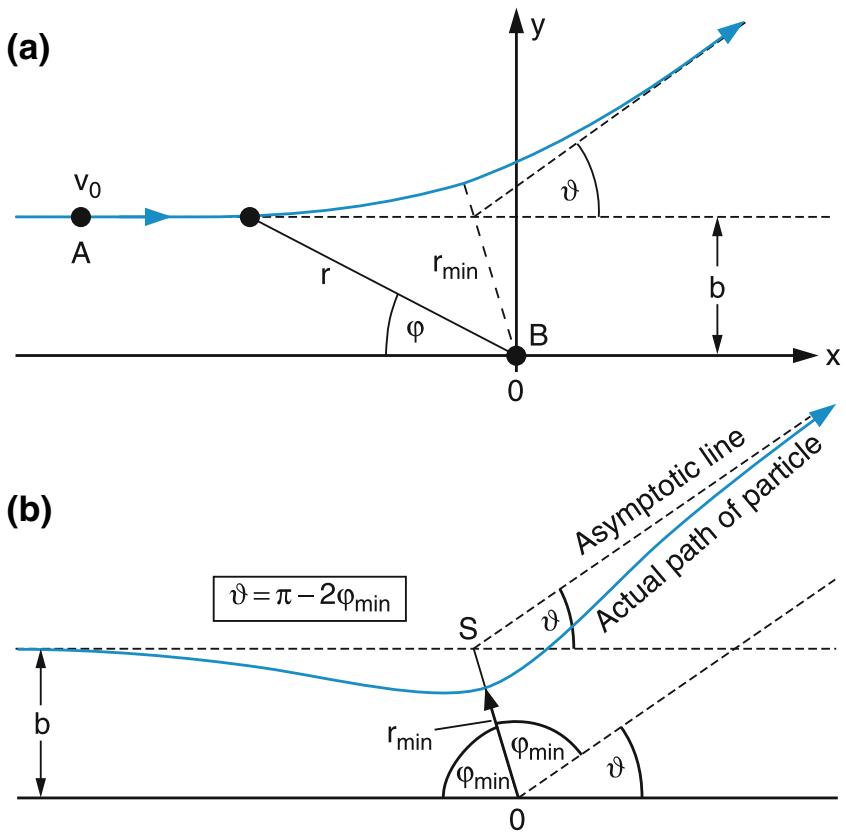
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$$= \mu r\dot{\varphi}\left(\boldsymbol{r} \times \hat{e}_t\right),$$

$$= \mu r^2\dot{\varphi} = \mu v_0 b,$$



The kinetic energy in center of mass frame

$$E_{\text{kin}} = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)$$
$$= \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2}.$$

The total energy

$$E_{\text{total}} = E_0 = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + E_{\text{pot}}(r) = \text{const.}$$

The derivatives of radii and angle are

$$\dot{r} = \left[\frac{2}{\mu}\left(E_0 - E_{\text{pot}}(r) - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}$$
$$\dot{\varphi} = \frac{L}{\mu r^2}.$$

Since for a spherically symmetric potential this path must be mirror-symmetric to the line OS

The relation the asymptotic scattering angle $\vartheta$ to the polar angle by

$$\vartheta = \pi - 2\varphi_{\min}.$$

This yields the relation

$$\varphi_{\min} = \int_{\varphi=0}^{\varphi_{\min}} \mathrm{d}\varphi = \int_{r=-\infty}^{r_{\min}} \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r$$

$$= \int_{r=-\infty}^{r_{\min}} (\dot{\varphi}/\dot{r})\,\mathrm{d}r = \int_{r_{\min}}^{+\infty} \frac{\dot{\varphi}}{\dot{r}}\mathrm{d}r.$$

The scattering angle in the CM-frame becomes

$$\vartheta(E_0, L) = \pi - 2\int_{r_{\min}}^{+\infty} \frac{(L/(\mu r^2))\,\mathrm{d}r}{\left[\frac{2}{\mu}\left(E_0 - E_{\mathrm{pot}}(r) - \frac{L^2}{2\mu r^2}\right)\right]^{1/2}}.$$

## The amount of the angular momentum

$$L = \mu r v \sin\varphi = \mu b v_0 \Rightarrow L^2 = \mu^2 b^2 v_0^2 = 2\mu b^2 E_0$$

## Therefore

$$\vartheta(E_0, b) = \pi - 2b \int_{r_{\min}}^{+\infty} \frac{\mathrm{d}r}{r^2 \left[1 - \frac{b^2}{r^2} - \frac{E_{\mathrm{pot}}(r)}{E_0}\right]^{1/2}}$$

## The lower integration limit $r_{min}$ is fixed by the condition

$$\dot{r}(r_{\min}) = 0$$

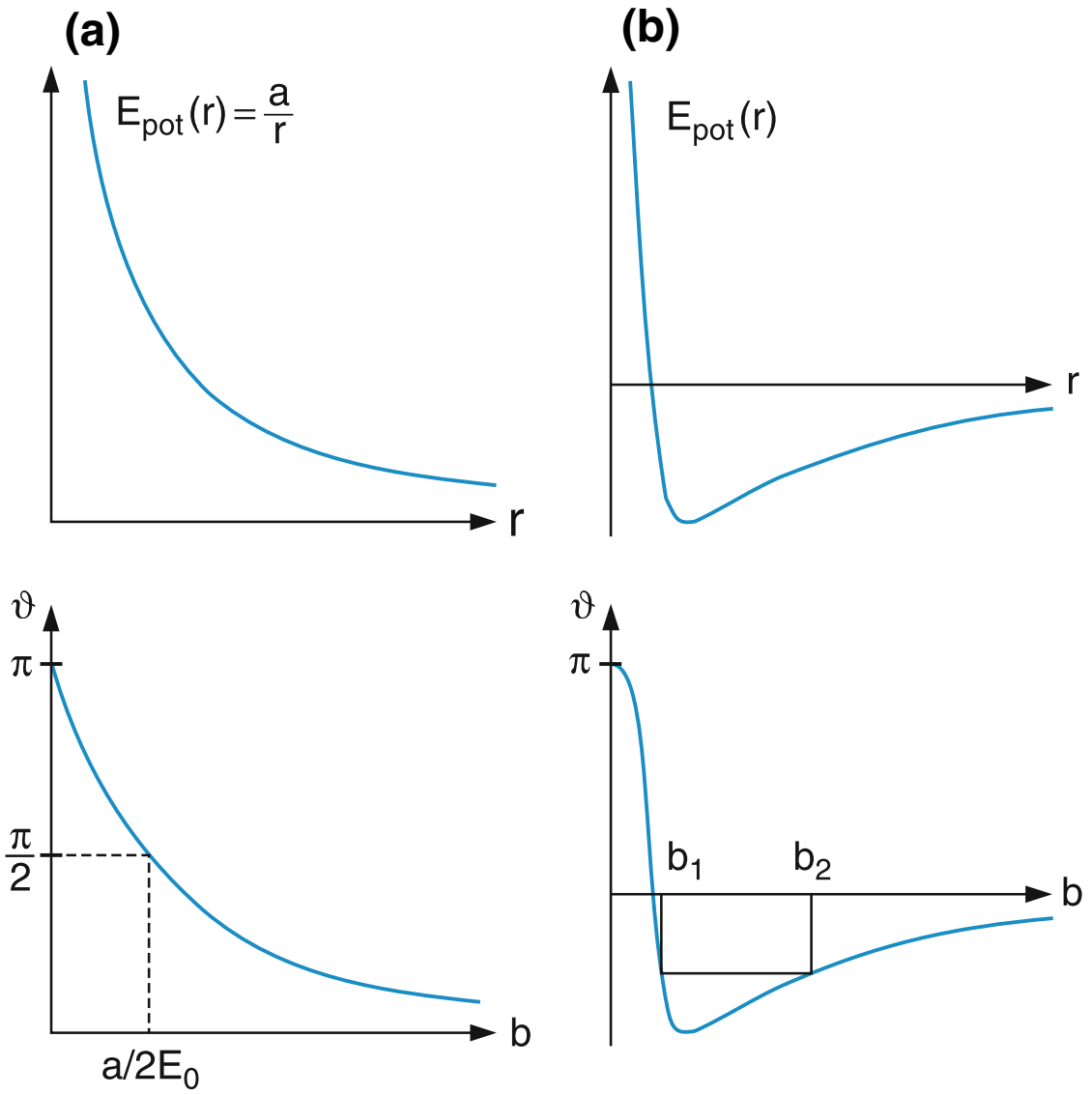
## which gives

$$r_{\min} = \frac{b}{\left[1 - \frac{E_{\mathrm{pot}}(r_{\min})}{E_0}\right]^{1/2}}$$

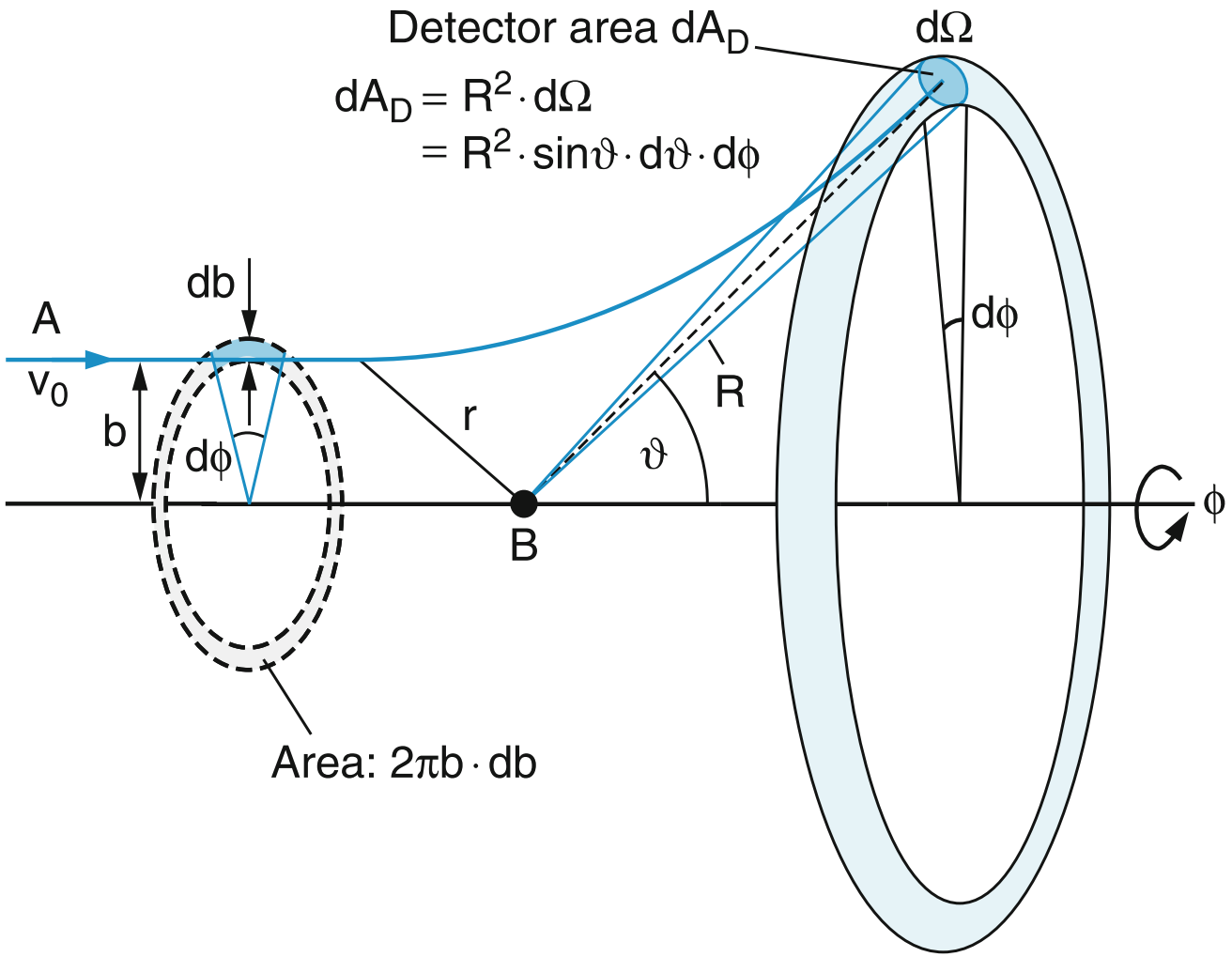
## The angular for Coulomb potential is

$$\vartheta = 2\cot^{-1}\left(\frac{4\pi\varepsilon_0}{qQ}\mu v_0^2 b\right)$$

$$E_{\mathrm{pot}} = \frac{qQ}{4\pi\varepsilon_0 r}$$


(a)
$E_{pot}(r) = \frac{a}{r}$
r
(b)
$E_{pot}(r)$
r
ϑ
π
$\frac{\pi}{2}$
b
$a/2E_0$
ϑ
π
$b_1$
$b_2$
b

# The cross section



# The cross section

Let us assume a parallel beam of incident particles A with particle flux density $\dot{N}_A = n_A v_A$ that passes through a layer of particles B in rest with density $n_B$.

All particles A passing through an annular ring with radius b and width db around an atom B are deflected by the angle $\theta$ and$\theta \pm d\theta/2$, assuming a spherically symmetric interaction potential.

The angular ring

$$d\dot{N}_A = \dot{N}_A \, dA = n_A v_A 2\pi b \, db$$

particles A pass per second

# The cross section

One particle B therefore scatters the fraction

$$\frac{\mathrm{d}\dot{N}_\mathrm{A}\left(\vartheta \pm \frac{1}{2}\mathrm{d}\vartheta\right)}{\dot{N}_\mathrm{A}} = 2\pi b\,\mathrm{d}b = 2\pi b\frac{\mathrm{d}b}{\mathrm{d}\vartheta}\mathrm{d}\vartheta$$

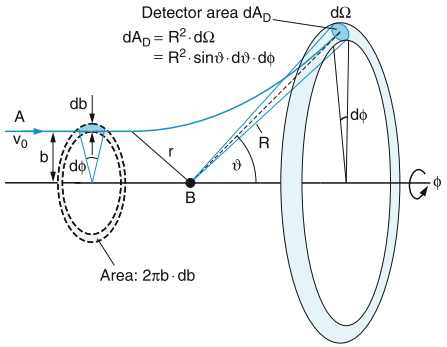
of all particles A, incident per second and unit area onto the target, into the range of deflection angles $\vartheta \pm \mathrm{d}\vartheta/2$.

The detector with area

$$A_\mathrm{D} = R^2\mathrm{d}\Omega = R^2\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\phi$$

receives the fraction

$$\frac{\mathrm{d}\dot{N}_\mathrm{A}(\vartheta,\phi)}{\dot{N}_\mathrm{A}}\frac{\mathrm{d}\phi}{2\pi} = b\frac{\mathrm{d}b}{\mathrm{d}\vartheta}\mathrm{d}\vartheta\,\mathrm{d}\phi$$



# The cross section

The fraction of all incident particles A, scattered by all atoms B with density $n_B$ in the volume $V = A\Delta x$ is then:

$$\frac{\mathrm{d}\dot{N}_\mathrm{A}(\vartheta, \mathrm{d}\Omega)}{\dot{N}_\mathrm{A}} = n_\mathrm{B} A \Delta x b \frac{\mathrm{d}b}{\mathrm{d}\vartheta} \mathrm{d}\vartheta \, \mathrm{d}\phi .$$

We define a differential cross section to be the ratio of scattered particles with per target and per unit solid angle to the number of incoming particles per unit area,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\text{scattered particles}}{\text{incident particles per unit area} \times \text{target particles}} \frac{1}{\mathrm{d}\Omega}$$

Therefore,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\dot{N}_\mathrm{A}(\vartheta, \mathrm{d}\Omega)}{\dot{N}_\mathrm{A} n_\mathrm{B} A \Delta x \mathrm{d}\Omega}$$

Therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = b\frac{\mathrm{d}b}{\mathrm{d}\vartheta}\frac{1}{\sin\vartheta}$$

with

$$\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\phi,$$

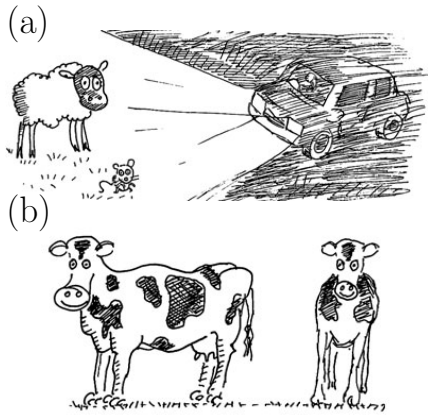



**Fig. 20.7** Scattering cross-sections. (a) $\sigma_{\text{sheep}} > \sigma_{\text{field mouse}}$. (b) $\sigma_{\text{cow, side}} > \sigma_{\text{cow, front}}$.

The relationship between b and θ for the Rutherford scattering yields

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

# The cross section

If $N$ incident particles strike a foil of thickness $t$ containing $n$ scattering centers per unit volume, the average number $dN$ of particles $\Omega$ scattered into the solid angle $d\Omega$ around $\Omega$ is given by

$$dN = Nnt\frac{d\sigma}{d\Omega}d\Omega$$

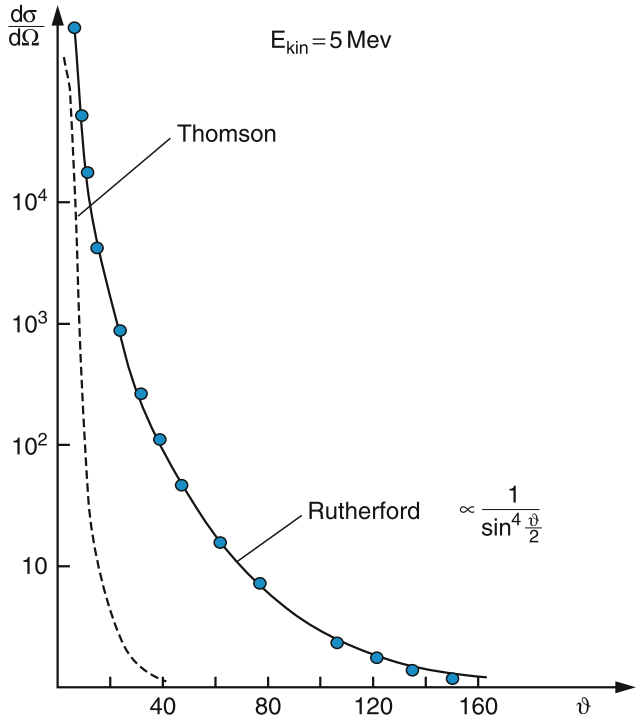
Therefore,

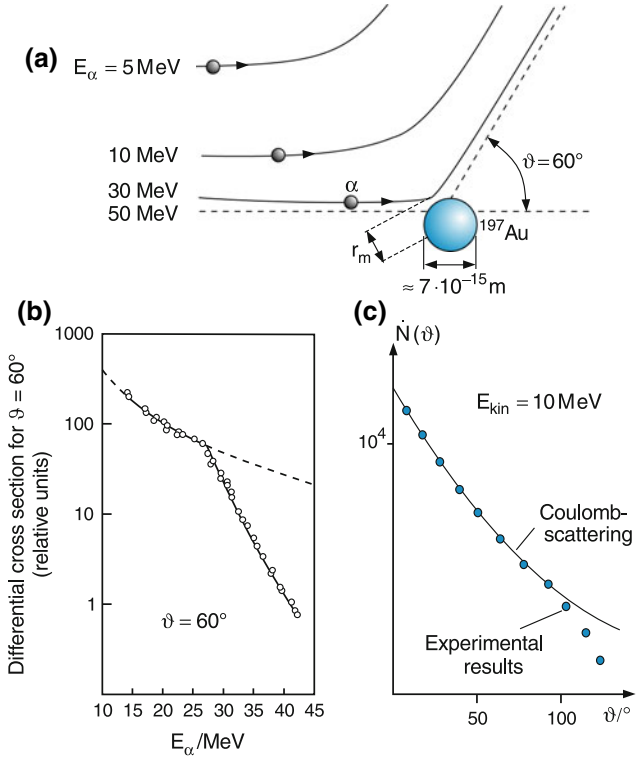
$$\frac{dN}{N} = nt\left(\frac{Z_1 Z_2 e^2}{16\pi\varepsilon_0 E}\right)^2 \frac{1}{\sin^4(\theta/2)}d\Omega$$

This is the Rutherford result explaining the Geiger-Marsden experiment

# The Geiger-Marsden experiment

Number of particles scattered at a given angle in Rutherford scattering is calculable and well understood, since it is defined by the well understood electromagnetic force.



# The key points

Impact parameter: b

Scattering angle: θ

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

Differential cross-section: the ratio of the number of particles scattered into an element of solid angle dΩ in the direction θ per unit area (unit 1 barn=$10^{-28}$ $m^2$)

The important quantities in Rutherford formula:

1. impact parameter
2. Scattering angle.
3. charges
4. Initial kinetic energy

# The total cross section

The integral scattering cross section is obtained by integration over dΩ, where the integration limits are $\vartheta(b = 0) = \pi$ and $\vartheta(b_{max}) = \vartheta_{min}$:

$$\sigma_{\rm int} = \int_{\Omega} \frac{{\rm d}\sigma}{{\rm d}\Omega} {\rm d}\Omega = \int_{\vartheta=\pi}^{\vartheta_{\rm min}} \int_{\phi=0}^{2\pi} \frac{{\rm d}\sigma}{{\rm d}\Omega} \sin\vartheta \, {\rm d}\vartheta \, {\rm d}\phi,$$

where $\vartheta_{min}$ is the smallest detectable deflection angle. The integration over $\varphi$ gives $2\pi$. Therefore, we have

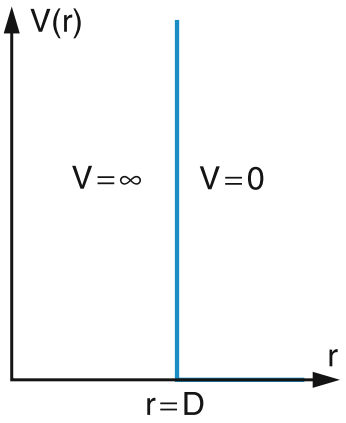
$$\sigma_{\rm int} = 2\pi \int_{\vartheta=\pi}^{\vartheta_{\rm min}} \frac{b}{\sin\vartheta} \left| \frac{{\rm d}b}{{\rm d}\vartheta} \right| \sin\vartheta \, {\rm d}\vartheta$$

$$= 2\pi \int_{b=0}^{b_{\rm max}} b \, {\rm d}b = \pi b_{\rm max}^2 .$$

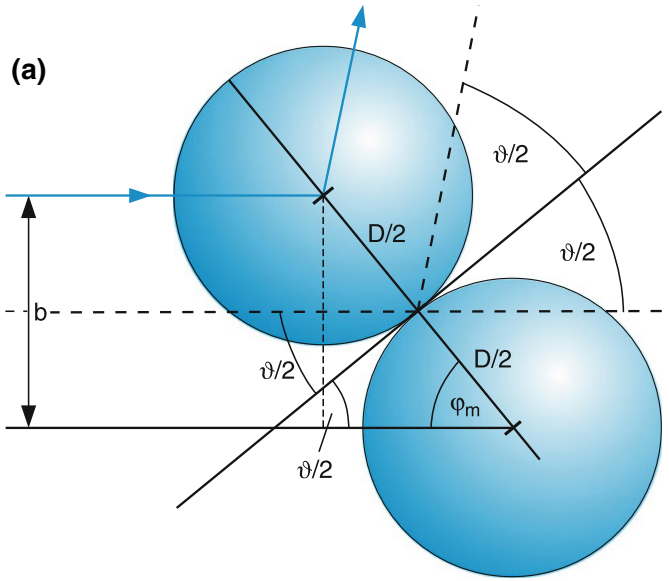
# The total cross section

Collisions of hard spheres A and B with equal diameters D. The potential energy in this case is:

$$E_{\text{pot}}(r) = \left\{ \begin{array}{ll} \infty & \text{for } r \leq D \\ 0 & \text{for } r > D \end{array} \right\}.$$




It is seen that at the closest approach $\sin\varphi_m = b/D$, which implies that a collision can only take place for $b \leq D$. For the scattering angle $\vartheta$ we find $\vartheta/2 = \pi/2 - \varphi_m$.



# The total cross section

The impact parameters for b ≤ D are therefore

$$b = D\sin\varphi_{\mathrm{m}} = D\cos(\vartheta/2).$$

Then the derivative db/d$\vartheta$ becomes

$$\left|\frac{\mathrm{d}b}{\mathrm{d}\vartheta}\right| = \frac{D}{2}\sin\vartheta/2$$

and the differential scattering cross section is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\vartheta}\frac{\mathrm{d}b}{\mathrm{d}\vartheta} = \frac{D\cos(\vartheta/2)D\sin(\vartheta/2)}{2\sin\vartheta} = \frac{D^2}{4}$$

$$\Rightarrow \sigma_{\mathrm{int}} = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = 4\pi\frac{D^2}{4} = \pi D^2.$$

The deflection function $\vartheta$(b) for hard spheres is

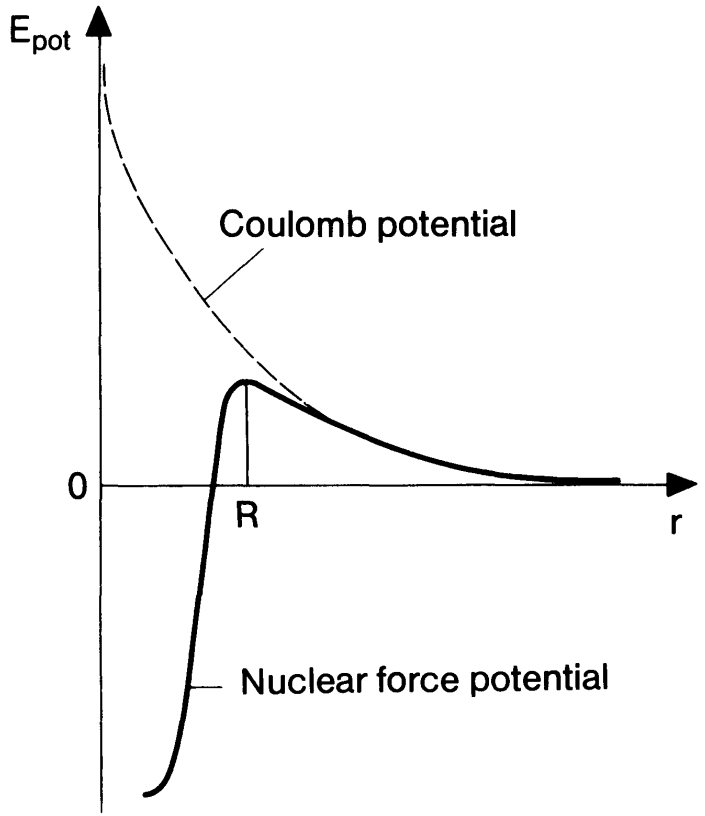
$$\vartheta(b) = \pi - 2\varphi_{\mathrm{m}} = \pi - 2\arcsin(b/2D).$$

# What is Meaning by Nuclear Radius?

The nuclear radius is defined as the distance at which the effect of the nuclear potential is comparable to that of the Coulomb potential

$$E = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{r}$$

$$r_{\min} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{E}$$




Empirical nuclear radius

$$R = 1.2A^{1/3}\ \text{fm}$$

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# The Physics of Atoms and Quanta

## 2.2, 2.4, 4.2, 4.4, 4.5, 4.6, 4.8

# Exercise 1

What is 0.134 kg of Li in moles, and what is the total number of atoms? (The mass of a Li atom is 6.941 u)

# Exercise 1

What is 0.134 kg of Li in moles, and what is the total number of atoms? (The atomic weight of Li atom is 6.941)

Solutions: There are

$$134/6.941 = 19.31 \quad \text{mol}$$

The total number of atoms is

$$19.31 \times 6.022 \times 10^{23} = 1.16 \times 10^{25}$$

# Exercise 2

The density of metallic strontium is 2.60 g $cm^{-3}$. Use this value to estimate the radius of the atom of Sr, whose atomic weight is 87.6.

# Exercise 2

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The density of metallic strontium is 2.60 g $cm^{-3}$. Use this value to estimate the radius of the atom of Sr, whose atomic weight is 87.6.

Solutions:

The molar volume of Sr is:

$$87.6/2.60 = 33.7 \text{ cm}^3\text{mol}^{-1}$$

The volume of each atom is

$$33.7/(6.022 \times 10^{23}) = 5.596 \times 10^{-23} \text{ cm}^3$$

The radius of Sr atom is

$$r = (3V/(4\pi))^{1/3} = 2.37 \times 10^{-10} \text{ m}$$

# Exercise 3

Calculate the impact parameter for scattering a 7.7 MeV $\alpha$ particle from gold at an angle of (a) 1° and (b) 90°.

# Exercise 3

Calculate the impact parameter for scattering a 7.7 MeV $\alpha$ particle from gold at an angle of (a) 1° and (b) 90°.

Solutions:

The $\alpha$ particle and gold nucleus have Z1 = 2 and Z2 = 79, respectively. For θ=1°,

$$b = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{2E}\frac{1}{\tan(\theta/2)} \qquad \frac{e^2}{4\pi\varepsilon_0} = 1.44\times10^{-9}\ \text{eV m}$$

$$= \frac{2\times79\times1.44\times10^{-9}}{2\times7.7\times10^{6}}\frac{1}{\tan(0.5\pi/180)} = 1.69\times10^{-12}\ \text{m}$$

for θ=90°,

$$b = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{2E}\frac{1}{\tan(\theta/2)}$$

$$= \frac{2\times79\times1.44\times10^{-9}}{2\times7.7\times10^{6}}\frac{1}{\tan(\pi/4)} = 1.48\times10^{-14}\ \text{m}$$

# Exercise 4

Rutherford found deviations from his scattering formula at backward angles when he scattered 7.7 MeV $\alpha$ particles ($Z_1$=2) on aluminum ($Z_2$=13). He suspected this was because the $\alpha$ particle might be affected by approaching the nucleus so closely. Estimate the size of the nucleus based on these data.

# Exercise 4

Rutherford found deviations from his scattering formula at backward angles when he scattered 7.7 MeV $\alpha$ particles ($Z_1$=2) on aluminum ($Z_2$=13). He suspected this was because the $\alpha$ particle might be affected by approaching the nucleus so closely. Estimate the size of the nucleus based on these data.

Solutions:

$$r_{\min} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{E}$$

$$= \frac{2 \times 13 \times 1.44 \times 10^{-9}}{7.7 \times 10^6} = 4.9 \text{ fm}$$

We find the sum of the $^4$He and aluminum nuclear radii to be about 5*10$^{-15}$ m.